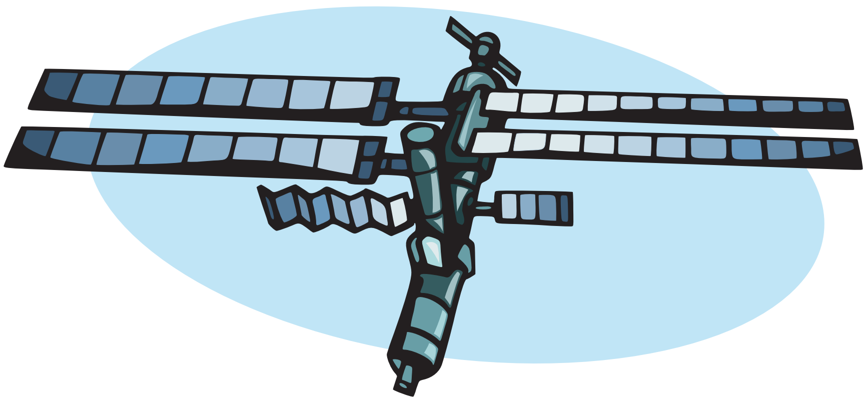
IT'S A PUZZLEMENT BY JON EVANS

Rockets Into Deep Space

Walter manages a research station deep in intergalactic space far from any stars or planets. The station routinely launches probes propelled by one or more identical ordinary expendable chemical rocket boosters. If two boosters are used and fired simultaneously, the probe is accelerated to a velocity 14 percent higher than if just one booster is used. Similarly, three boosters result in 20 percent higher velocity than just one.

Walter asks his assistant Wernher how many boosters would be required for double the velocity of using just one booster. Wernher calculates the answer using classical mechanics, appropriately ignoring any relativistic effects. Though the answer is accurate, Walter is disappointed. Walter then asks Wernher how many boosters would be required to double the velocity if, instead of firing them all at once, the boosters are fired sequentially one at a time, with each booster discarded immediately after it completes its burn. Again, Wernher gives a correct answer.

How did Wernher answer each of these questions?



A Simple Equation to Solve

Given the simple equation below, where the pattern of nested operations on the right hand side continues ad infinitum, what is the exact value of x?

$$\pi^{2/3}+e^{1/2} = \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+\ldots}}}}$$

Let

$$f(x) = \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+\ldots}}}}$$

Then the functional equation $f(x)^2 = 1+xf(x+1)$ holds for which $f(x) = 1+x$ is a solution. Since $f(x) = \pi^{2/3}+e^{1/2}$ then it should follow that $x = \pi^{2/3}+e^{1/2}-1$. However, we still need to prove uniqueness. The following sandwich inequality holds,

$$\sqrt{x\sqrt{x\sqrt{x\sqrt{\ldots}}}}$$

$$\leq \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+\ldots}}}}$$

$$\leq \sqrt{(1+x)\sqrt{2(x+1)\sqrt{4(x+1)\sqrt{\ldots}}}}$$

Note that

$$\sqrt{x\sqrt{x\sqrt{x\sqrt{\ldots}}}} = x^{2^{-1}+2^{-2}+\ldots} = x$$ and that

$$\sqrt{(1+x)\sqrt{2(x+1)\sqrt{4(x+1)\sqrt{\ldots}}}}$$

$$= ((x+1)^{2^{-1}+2^{-2}+\ldots})\sqrt{1\sqrt{2\sqrt{4\sqrt{\ldots}}}}$$

$$= (x+1)\sqrt{1\sqrt{2\sqrt{4\sqrt{\ldots}}}}$$

with $\sqrt{1\sqrt{2\sqrt{4\sqrt{\ldots}}}} = 2^A$ where

$A = \sum_{n=0}^{\infty} n\,2^{-n-1} = 1$ so that $\sqrt{1\sqrt{2\sqrt{4\sqrt{\ldots}}}} =$ 2 and hence $\sqrt{(1+x)\sqrt{2(x+1)\sqrt{4(x+1)\sqrt{\ldots}}}} = 2(x+1)$. The sandwich inequality can now be stated as

$$x \leq \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+\ldots}}}} \leq 2(x+1)$$

Since $3 < 3.79\ldots = x = \pi^{2/3}+e^{1/2} = \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+\ldots}}}} \leq 2(x+1)$

It follows that $x > ½$ and the sandwich inequality can be restated again as $½\,(x+1) < f(x) \leq 2(x+1)$, and we already can form another sandwich inequality $½+xf(x+1) < f(x)^2 < 2+xf(x+1)$ so that $½(1+x(x+1)) < f(x)^2 < 2(1+x(x+2))$ and consequently $\sqrt{½}(x+1) < f(x) < \sqrt{2}(x+1)$. Repeated iteration leads to $\sqrt[n]{½}(x+1) < f(x) < \sqrt[n]{2}(x+1)$. Since $\lim_{n\to\infty} \sqrt[n]{½} = \lim_{n\to\infty} \sqrt[n]{2} = 1$, the solution must be $f(x) = 1+x$ and $x = \pi^{2/3}+e^{1/2}-1$.

Solutions were submitted by David Andrist, Roger Bovard, Bob Conger, Mario DiCaro, Sidharth Garg, Akshar Gohil, Rob Kahn, Jerry Miccolis, Sean Moore, Anthony Salis, Dave Schofield, Alex Twist, Mark Woods and Michael Ziniti.

Know the answer? Send your solution to ar@casact.org.