Appendix to Explorations: In Praise of Value at Risk By Stephen Mildenhall

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We show that the crossed pairing provides the worst VaR pairing of X and Y. The proof is by induction on the number of points n being paired. We have already seen it is correct for $n = 2$, so assume it is true for any $n - 1$ points. Suppose X and Y have n points and that we have an optimal pairing producing the maximum minimum pairwise sum value. If the largest value of X is paired with the smallest value of Y, we can omit those two points, producing collections of $n - 1$ points where the optimal arrangement is crossed by induction. If the minimum paired sum of all n points in the optimal arrangement is the max of X plus min of Y, then all $n - 1$ remaining pairings must be greater than this value, but by induction the pairwise sum of the crossed arrangement of these $n - 1$ points is at least as large and hence also greater than the max of X plus min of Y. Conversely, if the minimum paired sum of the original n points is a different pair, then it will occur in the $n - 1$ remaining points and must equal the minimum of the crossed arrangement by induction. In either case, the crossed arrangement is optimal.

(0, 0) – (0, 14.200000000000001) node[above]X; (2, 0) – (2, 14.200000000000001) node[above]Y; (A1) at (0, 1) ; (A2) at (0, 2) ; (A3) at (0, 3) ; (A4) at (0, 4.5) ; (A5) at (0, 6) ; (A6) at (0, 8) ; (A7) at (0, 12) ; (B1) at (2, 5.214285714285714) ; (B2) at (2, 5.705372623492088) ; (B3) at (2, 6.261722393090486) ; (B4) at (2, 6.919728415189799) ; (B5) at (2, 7.7463472943680545) ; (B6) at (2, 8.892314581140788) ; (B7) at (2, 10.848415233348188) ; (A1) – (B7); (A2) – (B6); (A3) – (B5); (A4) – (B4); (A5) – (B3); (A6) – (B2); (A7) – (B1); (A1) – (B1); (A2) – (B2); (A3) – (B3); (A4) – (B4); (A5) – (B5); (A6) – (B6); (A7) – (B7); at (1, 3.107142857142857) ; at (1, 3.852686311746044) ; at (1, 4.6308611965452435) ; at (1, 5.709864207594899) ; at (1, 6.873173647184027) ; at (1, 8.446157290570394) ; at (1, 11.424207616674094) ; at (1, 5.924207616674094) ; at (1, 5.446157290570394) ; at (1, 5.373173647184027) ; at (1, 5.709864207594899) ; at (1, 6.1308611965452435) ; at (1, 6.852686311746044) ; at (1, 8.607142857142858) ;

On the other hand, suppose the largest value of X is not paired with the smallest value of Y. Then we can find two pairs: the largest value of X paired with a value y which is greater than the smallest value y_s of Y and a value x smaller than the largest value of X paired with the smallest value of Y. But if we simply swap these two paired values we will produce an arrangement with a greater minimum value (compare the case $n = 2$), contradicting our assumption that the arrangement was optimal. Hence this situation cannot occur. The worst VaR pairing for seven points is illustrated in Figure 2 [figtwo].