It's a Puzzlement by Jon Evans

Proof Outline for "More Refined Pricing"

1. Consider the space U of all functions $f\colon Z^d \to R^+$ that are equal to the average of their $2d$ nearest neighbors at all points in Z^d. Let e_i be the vector in Z^d that has value +1 in the ith coordinate, where $i = 1, \dots, d$ and value 0 in all other coordinates.
2. When X and Y are nearest neighbors in Z^d, $f(Y) \le 2d\, f(X)$. This inequality generalizes to $f(Y) \le (2d)^L\, f(0)$, where it is possible move from X to Y with L number of steps in the lattice. That is the number of $\pm\, e_i$ vectors that must be added to form $Y - X$.
3. Let X_t , $t = 0, \dots, +\infty$ be a random walk starting the origin with $X_0 = (0, \dots, 0)$ with $X_{t+1} = X_{t+1} \pm\ e_i$, where each of the $2d$ possible values for $\pm\, e_i$ is equally likely. $E[f(X_{t+1}) |\, X_t] = f(X_t)$ due to the harmonic constraint, and by induction $E[f(X_t)] = f(X_0)$, and also note that $E[f(X_t)^2] \le (2d)^{2t} f(X_0)^2$ since $f(X_{t+1})^2 \le (2d)^{2t} f(X_0)^2$
4. Let T be the non-negative integer valued random variable with density $P[T = t] = 2^{-3t-1} d^{-2t}$
5. $E[f(X_T)^2] \le f(X_0)^2 \sum_{t=1}^{\infty} 2^{-t} = f(X_0)^2$ and $p(Y) = P[X_T = Y] > 0$ for any $Y \in Z^d$,
6. Let $\| f \|_2 = E[f(X_T)^2]^{1/2} = (\sum_{Y \in Z^d} p(Y)\, f(Y)^2)^{1/2}$, and note that this forms a metric where $d(f, g) = \| f - g \|_2$. Let $V = \{f | f \in U\ \cap f(X_0) = 1\}$. Note, topologically V can be thought of as a subset of a compact space W that is homeomorphic to a countably infinite cartesian product of the closed interval [0, 1] since there is a maximum and minimum bound for any $f \in V$ at each point $X \in Z^d$.
7. Now we can find a specific $\bar{f}\ \in V$ so as to maximize $\| \bar{f} \|_2$. A sequence f_n where $\| f_n \|_2$ converges to the least upper bound of $\| f \|_2$ must exist and compactness of W ensures that there is a subsequence of these functions that also converges to a specific function $\bar{f}\ \in W$. This function $\bar{f}$ must also be harmonic since the subsequence must converge pointwise and similarly it must also have value 1 at the origin X_0. Everywhere $\bar{f} >$

0, since if $f(X) = 0$ for any point $X \in Z^d$, it would have to be that $f(X) = 0$ for all points $X \in Z^d$. Therefore $\bar{f} \in V$.

8. Consider that $\bar{f}(X) = \sum_{\pm,i}\left(\frac{f(X_0 \pm e_i)}{2d}\right) f_i^{\pm}(X)$ where $f_i^{\pm}(X) = \left(\frac{\bar{f}(X \pm e_i)}{\bar{f}(X_0 \pm e_i)}\right)$. Note $f_i^{\pm} \in V$ and in particular $\bar{f}(X_0) = \frac{\bar{f}(X_0 \pm e_i)}{\bar{f}(X_0 \pm e_i)} = 1$ implies that $\sum_{\pm,i}\left(\frac{\bar{f}(X_0 \pm e_i)}{2d}\right) = 1$. So $\| \bar{f} \|_2 \geq \| f_i^{\pm} \|_2$ and at the same time $\| \bar{f} \|_2 \leq \sum_{\pm,i}\left(\frac{f(X_0 \pm e_i)}{2d}\right) \| f_i^{\pm} \|_2$. So, it must be that $\| \bar{f} \|_2 = \sum_{\pm,i}\left(\frac{f(X_0 \pm e_i)}{2d}\right) \| f_i^{\pm} \|_2$, but this can only be the case if $f_i^{\pm} = C_i^{\pm}\bar{f}$, for some constants $C_i^{\pm} > 0$.

9. $f_i^{+} = C_i^{+}\bar{f}$ implies that $\bar{f}(X + e_i) = K_i\bar{f}(X)$ for some constants $K_i > 0$ and in turn that $\bar{f}(X - e_i) = \bar{f}(X)/K_i$. Restating the harmonic constraint $\bar{f}(X) = \frac{1}{2d}\sum_i (K_i + 1/K_i)\bar{f}(X)$. So $\frac{1}{2d}\sum_i (K_i + 1/K_i) = 1$. $K_i + 1/K_i$ has a minimum value of 2 only when $K_i = 1$, and these are the only values for K_i that satisfy the previous equation. Consequently $\bar{f}(X \pm e_i) = \bar{f}(X)$ and by induction this implies that $\bar{f}(X) = \bar{f}(X_0) = 1$, a constant function.

10. In general, for $g \in U$, we can create a corresponding $\bar{g} \in V$ as $\bar{g}(X) = \frac{g(X)}{g(X_0)}$. Now, $\| \bar{g} - 1 \|_2^2 = \| \bar{g} \|_2^2 - 1 \geq 0$ but at the same time time $\| \bar{g} \|_2^2 \leq \| \bar{f} \|_2^2 = 1$. So, $\| \bar{g} \|_2^2 = 1$ and consequently $\| \bar{g} - 1 \|_2^2 = 0$, but this can only happen if $\bar{g} = 1$, a constant function. Therefore, $g(X) = g(X_0)$, a constant function.