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Hanging Rope Solution
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Let b be the height of the shorter pole. Then $2b$ is the height of the taller pole, $3b$ is the length of the rope when stretched taut between the tops of the two poles and $\sqrt{(3b)^2 - b^2} = 2\sqrt{2}b$ is the distance between the poles.

For the first question, the maximum possible increase in the rope's length will occur when the rope forms a catenary with the bottom of the catenary at the top of the shorter pole. Assume this is at $(0, a)$ and the top of the taller pole is at $\left(2\sqrt{2}b, a\cosh\left(\frac{2\sqrt{2}b}{a}\right)\right)$. Then $a\cosh\left(\frac{2\sqrt{2}b}{a}\right) - a = b$ and thus $\cosh\left(2\sqrt{2}\frac{b}{a}\right) - 1 = \frac{b}{a}$. Solving this equation yields $\frac{b}{a} = 0.24057$. The length of the rope is $\int_0^{2\sqrt{2}b} \sqrt{1 + \left(\frac{d\, a\cosh\left(\frac{x}{a}\right)}{dx}\right)^2}\, dx = \int_0^{2\sqrt{2}b} \cosh\left(\frac{x}{a}\right) dx = a\sinh\left(2\sqrt{2}\frac{b}{a}\right)$. The ratio to the length when taut is $\frac{a\sinh\left(2\sqrt{2}\frac{b}{a}\right)}{3b} = \frac{\sinh\left(2\sqrt{2}\frac{b}{a}\right)}{3\frac{b}{a}} = 1.017$, so the maximum possible increase is 1.7%.

For the second question, the minimum possible increase in the rope's length will occur when the rope forms a catenary with the bottom of the catenary barely touching the ground. Assume this is at $(0, a)$, the top of the shorter pole is at $\left(-c, a\cosh\left(\frac{-c}{a}\right)\right)$ and the top of the taller pole is at $\left(2\sqrt{2}b - c, a\cosh\left(\frac{2\sqrt{2}b-c}{a}\right)\right)$. Then $a\cosh\left(\frac{-c}{a}\right) - a = b$ and $a\cosh\left(\frac{2\sqrt{2}b-c}{a}\right) - a = 2b$. Thus $\cosh\left(-\frac{c}{a}\right) - 1 = \frac{b}{a}$ and $\cosh\left(2\sqrt{2}\frac{b}{a} - \frac{c}{a}\right) - 1 = 2\frac{b}{a}$. Solving these equations yields $\frac{b}{a} = 1.15606$ and $\frac{c}{a} = 1.40271$. The length of the rope is $\int_{-c}^{2\sqrt{2}b-c} \cosh\left(\frac{x}{a}\right) dx = a\sinh\left(2\sqrt{2}\frac{b}{a} - \frac{c}{a}\right) - a\sinh\left(-\frac{c}{a}\right)$. The ratio to the length when taut is $\frac{\sinh\left(2\sqrt{2}\frac{b}{a} - \frac{c}{a}\right) - \sinh\left(-\frac{c}{a}\right)}{3\frac{b}{a}} = 1.461$, so the minimum possible increase is 46.1%.