Solution to Actuarial Review 45-4 Puzzle by Glenn Meyers

The equation for a (flexible) hanging rope is a classic one originally solved by Christiaan Huygens, Gottfried Leibniz, and Johann Bernoulli in 1691.

The equation for the path of a hanging rope is called a catenary and is given by the following equation.

$$y = a \cdot cosh(x/a) + C \tag{1}$$

The parameter a controls the horizontal scale of the curve. The parameter C controls the vertical placement of the curve, driven by the parameter a and the height of the poles. For the problems at hand, we need to set up some equations solve for a and C that match the conditions of the problem.

The plots of my solution are below. One should examine them as I sketch out my solution.

The poles have length 1 and 2, and the length of the taunt rope is 3, so the poles are $\sqrt{3^2 - 1^2} = \sqrt{8}$ apart.

Problem 1 is to find the maximum length of rope with the lowest hanging point still above the short pole.

- For the lowest point to be above the minimum, the shorter (left) pole should be located at $x \geq 0$. The maximum length occurs when the curvature is greatest, so the left pole should be placed at $x = 0$. For this point to have a height of 1, we have to set the constant $C = -a + 1$ in Equation 1.
- The right pole should be placed at the point $x = \sqrt{8}$. The height of the pole = 2. So the following equations should be satisfied.

$$a \cdot cosh(0) - a + 1 = 1 \tag{2}$$

$$a \cdot cosh(\sqrt{8}/a) - a + 1 = 2 \tag{3}$$

One can plug in $a = 4.156756$ to see that the above equations are satisfied.

Problem 2 is to find the length of the rope where its lowest point barely touches the ground.

- The lowest point of the catenary occurs at $x = 0$. For this point to be on the ground ($y = 0$) we set the constant C in Equation 1 equal to $-a$.
- Set the location of the left pole to be at $x = -L$ and the location of the right pole to be at $x = \sqrt{8} - L$. We then solve the following set of equations for L and a.

$$a \cdot cosh(-L/a) - a = 1 \tag{4}$$

$$a \cdot cosh((\sqrt{8} - L)/a) - a = 2 \tag{5}$$

One can plug in $a = 0.8650167$ and $L = 1.2133724$ to see that the above equations are satisfied.

Having determined the parameters of the catenaries, we now turn to calculating the length of the ropes for each problem. To do this we use the standard formula for arc length.

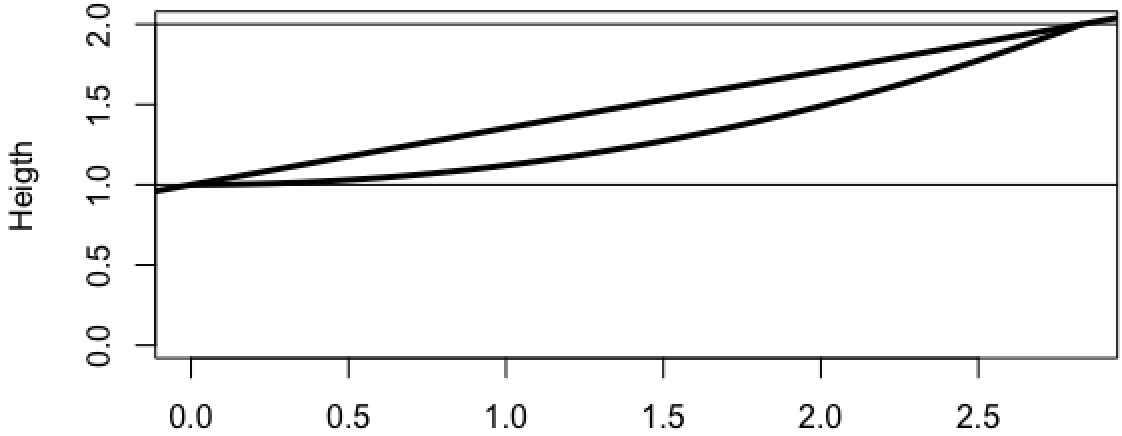
$$\int_{Left}^{Right} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{Left}^{Right} \sqrt{1 + sinh^2\left(\frac{x}{a}\right)} dx \tag{6}$$

Plugging in the parameters for Problem 1 gets an arc length of 3.051796 for a 7.90% increase over the length of the taunt rope.

Plugging in the parameters for Problem 2 gets an arc length of 4.3835143 for a 54.98% increase over the length of the taunt rope.

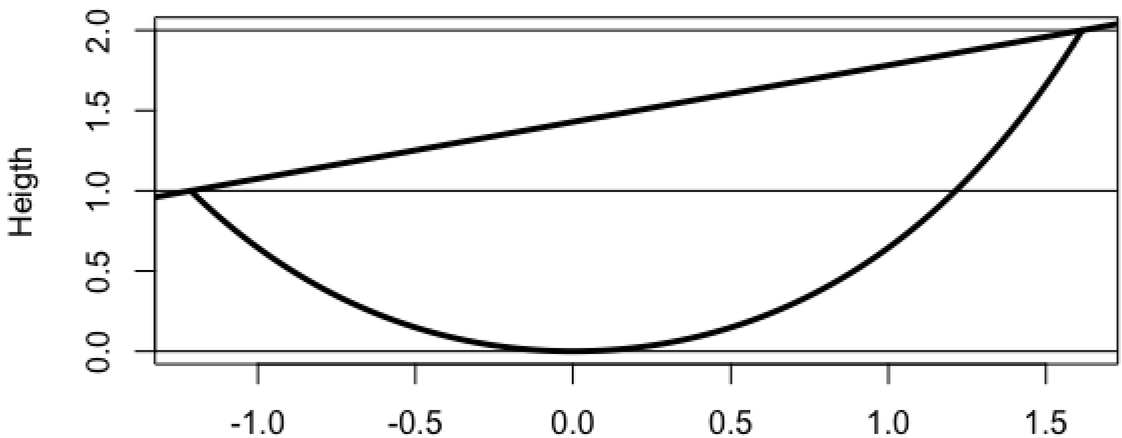
As for the numerics, I used the R “optimizer” function to solve Equation 3. I used the R “optim” function to solve Equations 4 and 5. I used the R “integrate” function to get the arc lengths. The R script is attached.

Problem 1



x
Increase in Rope Length = 7.9 %

Problem 2



x
Increase in Rope Length = 54.98 %