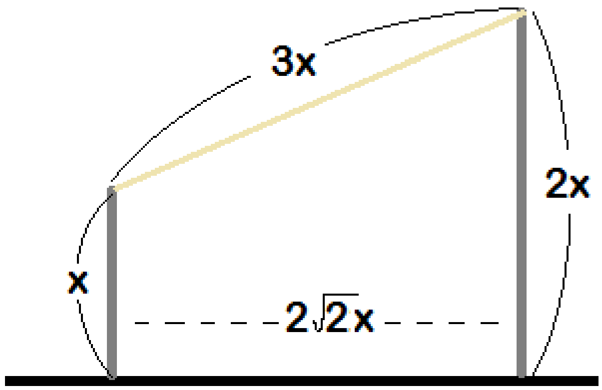
Hanging Rope - Submission by Hannah Park

Two ends of a rope are fixed to the tops of two poles standing straight above a float ground surface. The poles are unequal in height, with one pole being twice as tall as the other. When the rope is taut and forms a straight line between the tops of the poles, then its length is equal to the sum of the heights of the two poles.

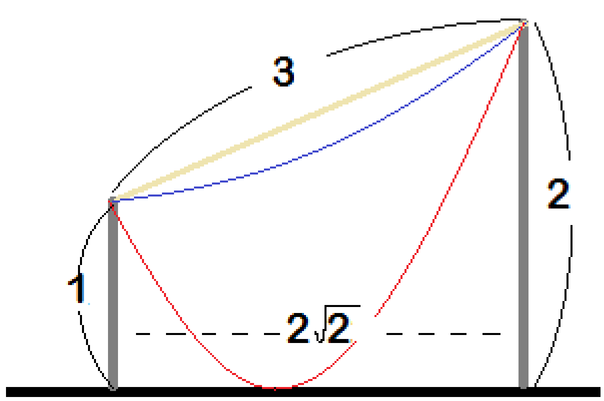


Since the two poles are upright, we can also find that the distance between the poles are equal to 2sqrt(2) times the height of the shorter pole.

1) What is the approximate maximum possible percentage increase in the rope's length, relative to its length when taut, so that its lowest hanging point will still be no lower than the top of the shorter pole?
2) What is the approximate minimum percentage increase in the rope's length, relative to its length when taut, so that its lowest hanging point just barely touches the ground?

Without loss of generality, we can ignore the variable x to simplify the case. (Since both questions are asking about the ratio of rope's length, this will result with the same final answer.)

Although it was not explicitly stated, we are making the following assumptions: the rope is inelastic, completely flexible, has uniform mass/density and the diameter is negligible.



The free hanging rope forms a curve called "catenary"

Solution

1) The maximum increase occurs when the short pole coincides with the axis of symmetry of the catenary curve. In other words, the vertex sits at the tip of the shorter pole.

Using:

$d = \frac{L^2-h^2}{2h}\ln(\frac{L+h}{L-h})$

where d = length between two poles and h = difference in pole height,

We have: $2\sqrt{2} = \frac{L^2-1}{2}\ln(\frac{L+1}{L-1})$ and $L \approx 3.051798$,or approximately **1.7% increase** is the maximum increase for problem #1.

2) The minimum increase occurs when the catenary curve is tangent to the ground. That is, the vertex sits at the ground somewhere between the two poles.

Using:

$$d = \frac{(L^2-a^2)(a+2h) - 2L\sqrt{h(a+h)(L^2-a^2)}}{a^2}\operatorname{atanh}(\frac{a^2}{L(a+2h) - 2\sqrt{h(a+h)(L^2-a^2)}})$$

where d = length between two poles,

h = difference in pole height,

a= difference in height between shorter pole and ground

We have :

$$2\sqrt{2} = \frac{(L^2-1)(3) - 2L\sqrt{2(L^2-1)}}{1}\operatorname{atanh}\left(\frac{1}{3L - 2\sqrt{2(L^2-1)}}\right)$$

and $L \approx 4.38358$,or approximately **46.1% increase** is the minimum increase for problem #2.

* Derivation of formulas can be found in
http://members.chello.nl/j.beentjes3/Ruud/catfiles/catenary.pdf

*$\operatorname{atanh}(x) = \frac{1}{2}\ln(\frac{1+x}{1-x})$

**Calculation performed on www.wolframalpha.com*