Lawn Mower Geometry

Solution by Bob Conger

First, we can determine a formula for a lower bound on the theoretical minimum time to mow a given area, without any time lost to double-mowing or stopping. An upper bound on the area of grass the lawn mover can cut is $A_{UB} = \pi \frac{D^2}{4} + Dx$, where D is the diameter of the circle of the cutting blade in motion and x is length of a path the mower moves along. Furthermore, if the mower moves at velocity v for time t then $A_{UB} = \pi \frac{D^2}{4} + Dvt$. Consequently, this implies that given a fixed area to mow a lower bound on the time required is $T_{LB} = Max\left[0, \frac{A-\pi\frac{D^2}{4}}{Dv}\right]$. In the particular case of this problem, since $D = 3$ ft and $v = 10$ ft/sec, $T_{LB} = Max\left[0, \frac{A}{30} - 0.235619\right]$ seconds. Later, we will compare actual cutting pattern times (including time lost to double-mowing and stop turns) to this theoretical lower bound.

The statement of the puzzle is not entirely clear about turning times. We have interpreted that a "pivot" turn of any angle from 0 to 180 degrees (such as we might encounter when navigating a triangle) can be accomplished in 4 seconds. On the other hand, we decided to utilize two different assumptions for "U-turns" that involve a combination of: a pivot turn of some angle (0 to 180 degrees), then a shift of some distance to reach a parallel path, and finally another pivot turn (to complete the remainder of the 180 degrees). We used 4 seconds as the optimistic assumption for this complete U-turn maneuver, while we used 4 + (shift distance/10) + 4 seconds as the pessimistic assumption.

It also is worth noting that since the mower can turn at 45 degrees per second, the tightest circular path it can make, without slowing, has radius $\frac{80}{2\pi} = 12.7324$ feet. Accounting for the diameter of the cutting blade, this would leave an inner circle of radius 11.2324 feet uncut, if the mower were to cut along this path. So, the shapes given (rectangle, triangle, parallelogram) don't really lend themselves to being efficiently cut through continuous turning, but rather through pivot turns and "U-turns" as discussed above.

Macro Strategy and Nomenclature

Regardless of how we interpret the turning parameters, turns are very time-consuming, based on the parameters in the puzzle. Therefore, we want as few turns as possible. This implies that a plausible candidate solution can be formulated by identifying the smallest cross section of the property, and mowing in a series of parallel back-and-forth strips that are perpendicular to the line of that cross section. For convenience, we refer to the small cross section as "north-south", "up-down", or as the y-axis. The dominant mowing direction for this candidate solution we refer to as "east-west", "left-right", or as the x-axis. When we use x-y coordinates, the point (0,0) is the southwest corner of the property, and the distance units are feet unless otherwise specified.

Other factors to keep in mind are minimizing the amount of time spent double-mowing (cutting over property areas that already have been cut), and minimizing the area cut on neighboring properties (though some will be inevitable, given a circular blade cut and non-rounded property corners and edges). And, of course, we need to assure that we have not failed to cut any areas within the property boundaries. In all of our strategies and calculations, we allowed the mower to begin and end at any locations and pointed in any directions; we did not attempt to designate a mandatory “home” location for the mower.

A disadvantage of the simple east-west back-and-forth, or “Shuttle” solution is the number of time-consuming U-turns required. Another disadvantage of the Shuttle solution is that we must do some extra work to avoid unmowed crescents inside the edge of the property at the end of each east-west mowing strip (due to having a circular blade cut). After some experimentation, we concluded that it is much more efficient to deal with these by extending the length of each east-west cut by a foot or two (mowing part of the neighbor’s lawn), which adds just a fraction of a second for each mowing strip -- rather than making separate north-sound cuts at the property boundaries, which cost several expensive turns.

When a triangle is involved (either on its own, or embedded in a trapezoid), we will want to explore solutions that mow around the inside of the perimeter of the property and spiral inward, as the “Spiral” solution requires only 3 simple pivot turns to make a cycle (versus 2 complex U-turns in the horizontal Shuttle solution); another potential advantage of a Spiral strategy for triangles is that the northernmost tip of the property (the apex of the triangle) is reduced by more than 3 feet with each mower cycle around the triangle due to the angles at which the property lines come together, so that fewer cycles may be required to complete the job.

A significant disadvantage of the Spiral strategy is that it leaves an unmowed crescent at the outer bend of each turn. We explored resolving this issue by overlapping the sequential cycles on the crescent, but that resolution increases the number of cycles required, thus increasing mowing distance as well as turns. Instead, noting that the unmowed crescents will be in an orderly pattern lined up between each vertex/corner and the center of the property, we considered adding 3 (for triangles) or 4 (for quadrilaterals) repair cuts specifically to mow over the unmowed crescent areas. The trick is to organize these repair cuts into the job in a way that we don’t spend an unnecessary amount of extra time (distance and turns) getting to and between the repair cuts.

We did not find any faster mowing times among more complex patterns, which is due to the extra mowing distance and turns involved in such patterns. But our failure to find faster solutions does not constitute a proof; other faster solutions may be possible.

Rectangle

For the 100x50 rectangle, we will treat one of the 100-foot sides as the base, and the 50-foot dimension as the altitude of the rectangle. A good basic strategy should be 17 mowing strips, centered 3 feet apart along the longer 100-foot direction, and connected by 16 U-turns. The mower starts near the southern end of the western property line, and travels 100 feet back and forth between the western property line and the eastern property line on each pass; 170 seconds in total would be required for the long strip cuts (plus time for turns, see below). Illustration R1 shows this basic strategy; we can see that there is some waste, both in the extra one foot that is cut all the way along the northern property line, and in the scallops of extra cutting *outside* the eastern and western boundaries that the mower creates in the process of making sure not to leave any unmowed area *inside* the boundaries.

We can improve the basic Shuttle strategy for mowing the rectangle by utilizing the extra 1 foot of mowing to accomplish something productive <u>within</u> the property boundaries (rather than wasting it outside the northern boundary). Specifically, we can use this extra mowing capacity to create overlaps in the pairs of adjacent mowing strips, which will allow us to stop and turn the mower slightly before it reaches the eastern and western property boundaries. Examining mowing strips 1 and 2 in Illustration R1, we can see that because these two strips just barely touch in the basic solution, it is necessary to have the center of the mower reach the western properly line in order that nothing is left unmowed inside the western property line. However, if these two mowing strips were to overlap at their western ends, the two final mowing circles would overlap and allow mowing on strips 1 and 2 to stop short of the western property line. On the other hand, no overlap is needed at the eastern end of strips 1 and 2, as the mowing activity to transit from strip 1 to strip 2 already covers this area. The same description applies to strips 3 and 4; and so on. The situation is reversed for strips 2 and 3, where overlap of these two strips at their <u>eastern</u> ends would allow shortening the mower approach to the eastern property line; but no overlap is needed at their western end. This alternating pattern continues for the remaining mowing strips. In addition, we need to overcut both ends of the northern and southern boundaries by half of the amount of the inter-strip overlaps so as to assure that we cut all the way into the property corners after we shorten the mower cuts at the eastern and westerns ends of strips 1 and 17. (Note that it is more efficient to allocate the total overlap evenly across the several overlap locations in the rectangle, rather than to focus the overlap in a small number of locations. This is a consequence of the fact that a larger single overlap produces diminishing returns. For example, an overlap of .05 foot will save 0.38 feet of cutting distance, while doubling that overlap to .10 only increases the savings in cutting distance by an additional 0.15 feet.)

Thus, we will have 8 + 2 halves = 9 overlaps on each (east and west) side of the property, or 1/9 of a foot for each overlap, as shown in illustration R-2. Where the strips overlap, the mower centers of the two strips will be (3 – 1/9) feet apart at the eastern or western end of each pair of mowing strips. And mowing strips 1 and 17 will overlap the southern and northern boundaries by 1/18 foot (i.e., the center of the mower will travel parallel to these boundaries, but 26/18 feet inside the boundaries).

Applying the Pythagorean Theorem, these overlaps allow us to begin and end the strip cuts with the blade cutting circle center inside from the eastern and western property lines by $\sqrt{1.5^2 - \left(1.5 - \frac{1}{18}\right)^2}$ = 0.404451 feet at both ends. The nine odd-numbered strips have a length of 99.1911 feet. The eight even-numbered strips extend the same distance east-to-west, but also are 0.1111 feet more southerly at their western end than at their eastern end, so have a slightly greater length of 99.1912 feet. The seventeen long linear cuts thus total 1686.25 feet, and require 168.63 seconds of cutting. Turns cost either 4 seconds or 8.3 seconds (the mower still shifts 3 feet at each U-turn, due to the alternating placements of the overlaps) for a total turn time between 64 and 132.8 seconds for the 16 turns. The total time range is between 232.63 seconds and 301.42 seconds, with 28% to 44% of that time spent on the turns.

Rectangle illustrations

Illustration R1: Rectangle solution: Basic back-and-forth

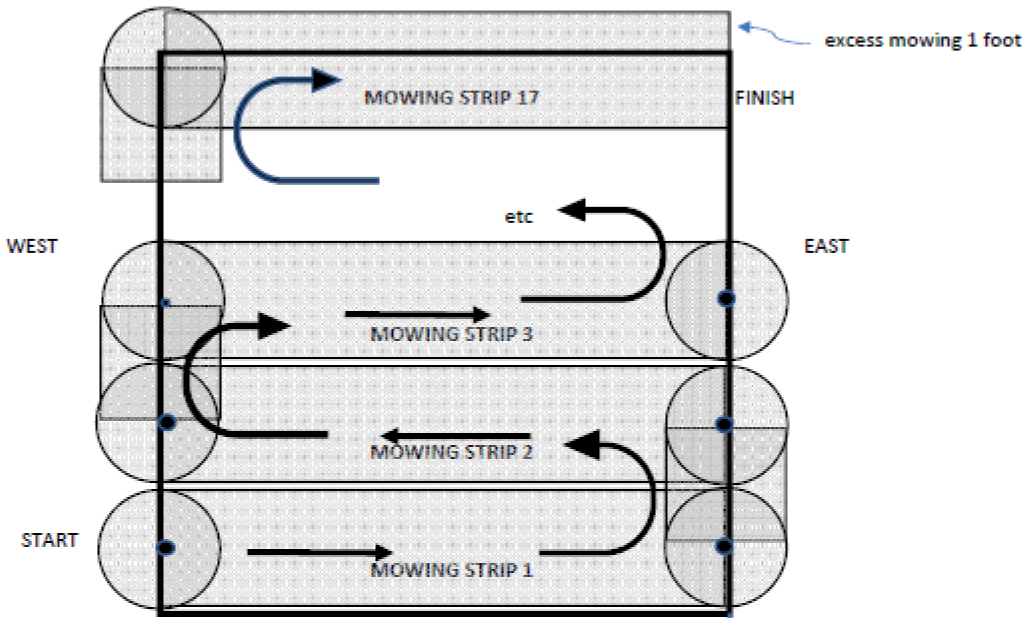
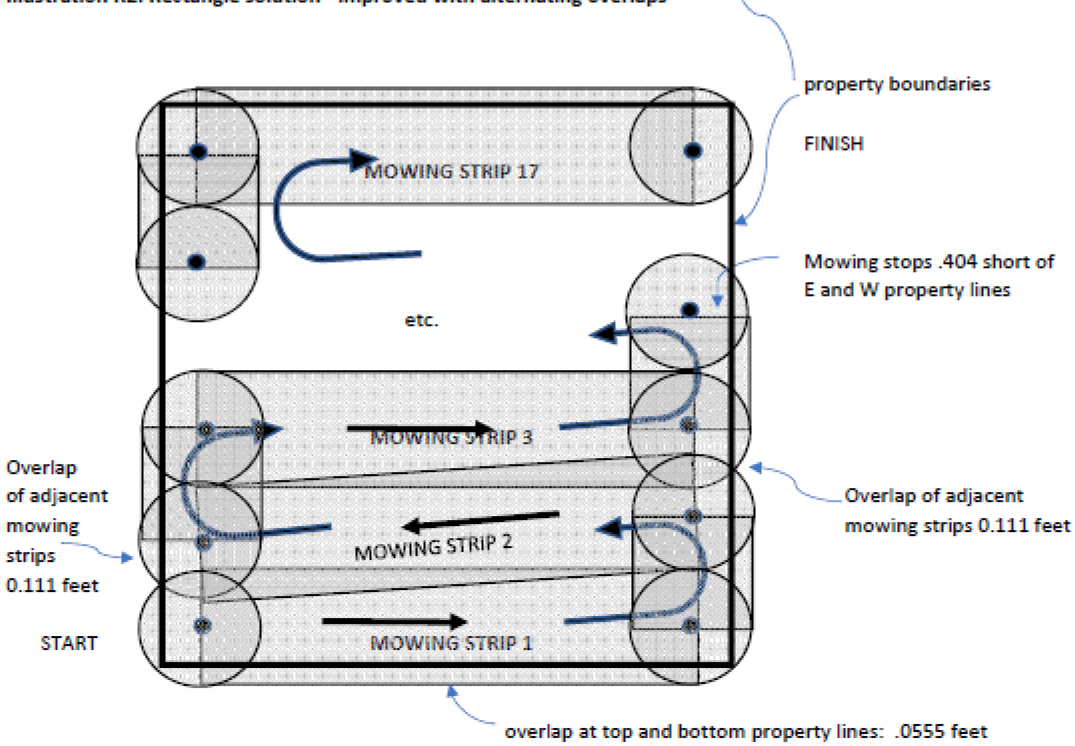


Illustration R2: Rectangle solution - improved with alternating overlaps



Triangle

With triangle sides of 100 and 50 feet joined by a right angle, the hypotenuse (length 111.803 feet) is the longest linear stretch, and the altitude of 44.721 feet from the hypotenuse to the opposite right angle vertex is the shortest altitude. We will treat the hypotenuse as the base (with vertices at (0,0) and (111.803,0)), and it is the southern boundary of the triangular property. For ease of description (and no loss of generality), we place the 50-foot side to the west, creating a slope of 2 on that side; and the 100 foot side to the east, creating a slope of 0.5 on that side (slopes cited as absolute values). The upper vertex is at (22.361, 44.721).

Triangle – Shuttle Strategy

An initial basic strategy for the triangle, similar to the initial basic Shuttle strategy for the rectangle, is to start with a mowing strip of 111.803 feet from west to east along the southern property line, matching the long edge of that grass strip. The next grass strip, shifted up 3 feet, has a long southern side that is 7.5 feet shorter than the triangle's base, so we could mow that with a east-to-west cut of 104.302 feet. Successive mowing strips each reduce by 7.5 feet in length. The final mowing strip, #15, has a length of (111.803 – (14 x 7.5)) = 6.803 feet, and extends 0.279 feet above the top of the triangle. The total mowing length of the horizontal cuts is 889.55 feet. Illustration T1 shows this strategy.

We can make two incremental improvements to this strategy. Illustration T2 shows a tighter placement of the end points of the mowing strips. The eastern ends of the mowing strips progress westward by 12 feet, then 0 feet, then 12 feet… rather than a shift of 6 feet for each strip. Similarly, the western ends of the mowing strips progress eastward by 0 feet and 3 feet alternately, rather than a shift of 1.5 feet for each strip. The lengths of the mowing strips are 111.803, 98.303, 95.303, 83.303, and continue shrinking by 3 and 12 feet; the final mowing strip is 5.303 feet. The total mowing length of the horizontal cuts is 837.051 feet. (However, this is less of an improvement than it initially appears, because the turn times will increase due to the larger shift distance that needs to be incorporated in the middle of each U-turn.)

As noted above, we have excess mowing of 0.279 feet above the top of the triangle. As with the rectangle, we can use this extra mowing capacity to create overlaps between mowing strips and save a tiny bit of cutting time. However, unlike the rectangle, where it is optimal to allocated the total overlap evenly across the different pairs that are to be overlapped, it is preferable to allocate most of the overlap to the higher (further north) mowing strips in the case of the triangle. Why? Let's examine the effect of how we place a single element, .02 feet, of overlap. Consider first what happens if we allocate the .02 of overlap to the eastern end of mowing strips 2 and 3. This overlap does not allow shortening mowing strip 2 because its length is determined by the length of the grass strip at its upper boundary. Mowing on strip 3 gets to stop .244 feet = SQRT ((1.5^2) – (((1.5-.02)^2))) short of the end of the grass as a result of the overlap. Since we have shifted the eastern end of mowing strip 3 by .02 feet to the south, the eastern ends of all the remaining mowing strips (4 through 15) each also shift .02 feet to the south, which means that these 12 mowing strips each gets longer by .04 feet (due to the slope of 0.5), or a total of .48 additional feet. So, when we allocate the .02 overlap to the eastern ends of mowing strips 2 and 3, we <u>add</u> more mowing time as a result of subsequent strips being lengthened (.48 feet) than we save by the overlap allowing the mowing strip 3 to be reduced by .244 feet. By contrast, if we allocate this same .02 of overlap to the eastern end of mowing strips 12 and 13, the savings for mowing strip 13 is the same as above (.244 feet reduction), while the increased mowing effort due to subsequent grass strips lengthening is only .08, because only two more northerly mowing strips are affected. In general terms, we should allocate most of the overlap to the more northern strips. To perform this allocation process, we constructed a function to estimate the marginal gains (mowing distance reductions due to overlap) and losses (mowing distance increases due to subsequent mowing strips shifting southward and getting longer) associated with allocating the excess mowing capacity across the fifteen mowing strips in different proportions, and selected the “western” allocation that equalizes the marginal net impact of the amounts allocated to each of the western ends of the mowing strips (but not allocating so much that the marginal net impact drops below zero) and similarly selected the “eastern” allocation that equalizes the net marginal impact of the amounts allocated to each of the eastern ends of the mowing strips. The resulting overlap allocations (feet) are as follows, shown as the amount of overlap between strip N and the next more southerly strip N-1:

Mowing Strip #	Overlap allocated (feet)	
	Western end	Eastern end
15	.09	.24
14	.06	
13		.02
12	.04	
11		.01
10	.03	
9		<.01
8	.02	
7		<.01
6	.01	
5		<.01
4	.01	
3		<.01
2	.01	
1	.01	<.01

It is interesting to note the very different allocations of the total overlap to the eastern versus the western ends of the mowing strips. Recall that the slope of the eastern side of the triangle is 0.5, while the slope of the western side is 2.0 (absolute values). With these slopes, the mowing lengths on the eastern side expand 4x what occurs on the western side, given a comparable southward shift in a moving strip. Thus, on the eastern side, it is much more important to avoid shifting a low-numbered mowing strip downward, as that shift will cause a more greater increase in mowing lengths as compared to the savings from introducing (or increasing) the time savings effect of an overlap. Thus, on the eastern end of the property, most of the overlap gets allocated to the high-numbered mowing strips. On the western end of the property, the allocation also is noticeably non-equal, but not nearly as much so as on the eastern end.

All this work to allocate a few inches of overlap ends up saving about 5 feet of moving, or 0.5 seconds of project time, reducing our mowing distance (not including turns) to 834.13 feet, or 82.41 seconds. However, the intellectual energy spent here also will be useful when we come to the trapezoids.

The U-turns in this strategy involve significant shifts between the pivots, averaging 4.02 feet of shift on the western end of the property (approximately 3 feet eastward and 3 feet northward), and 12.25 feet on the eastern end of the property (approximately 12 feet westward and 3 feet northward) due to the different slopes, see illustration T2, for an overall average shift of 8.14 feet. Thus, the average time per turn is 4 seconds (optimistic assumption) or 8.814 seconds (pessimistic), and the total turn time is 56 to 123.39 seconds.

Our overall project time with this refined Shuttle strategy is 834.13 feet/10 = 83.41 seconds of east-west mowing time, plus the 14 turns, for a total project time of = 139.41 to 206.80 seconds, of which 40% to 60% is spent on turns.

Triangle – Spiral Strategy

We also explored a completely different mowing strategy for the triangle, which we refer to as the "Spiral strategy." This strategy begins with a cut all the way around the triangle just inside the perimeter, and then continues with ever-decreasing nested triangles. Six circuits of the triangular shape are required. However, as noted earlier, this strategy leaves an unmowed crescent at the outside of each of the pivot turns, and thus requires three corrective cuts (one from each vertex to the center of the triangle) to clear out those trouble spots. For efficiency reasons, we conduct two of the corrective cuts and the beginning of the project, and one at the end. Illustration T3 shows a schematic map of this mowing strategy.

The perimeter of the triangle is 261.803 feet. The first circuit of the spiral, during which the mower center travels 1.5 feet inside the triangle, is a travel distance of 241.24 feet (measured along the path of the center of the mower blade), or 20.6 feet less than the perimeter. The second circuit travels 200.12 feet, and each subsequent circuit requires 41.12 fewer feet of mowing, for a total travel distance of 830.58 feet (and 17 pivot turns) for the six circuits. The sixth circuit leaves a small triangle (6.4 foot base, 2.6 foot altitude), which we are able to mow during the course of our three clean-up mowing strips. The three clean-up mowing strips (between the three vertices and the center of the triangle), total 144.4 feet of travel, and require 3 extra pivot turns (and a careful surveyor to get the paths aligned properly). The pivot turns require only 4 seconds each, so our total mowing time is 97.5 seconds, plus 80 seconds for turns, for a total project time of 177.5 seconds, of which 45% is spent turning.

Note that as anticipated, the Spiral strategy requires 6 cycles including 17 simple pivot turns, while the Shuttle strategy requires 7.5 "round trips" including 14 complex turns (each of which incorporates two pivot turns, and a distance shift). The Spiral strategy also must append a clean-up task (with 3 extra turns), of course.

This Spiral strategy time result is approximately midway between the two outcomes of the enhanced Shuttle mowing strategy. If the turn times in the Shuttle strategy are less than 6.72 seconds, the enhanced Shuttle strategy wins; if the Shuttle strategy turn times are slower than 6.72 seconds, the Spiral strategy outperforms the Shuttle strategy.

Triangle illustrations

Illustration T3: Triangle solution: Spiraling triangles strategy

Path description

Begin in lower left corner, travel to center of triangle

Turn left and proceed to a point slightly to the left of the top vertex

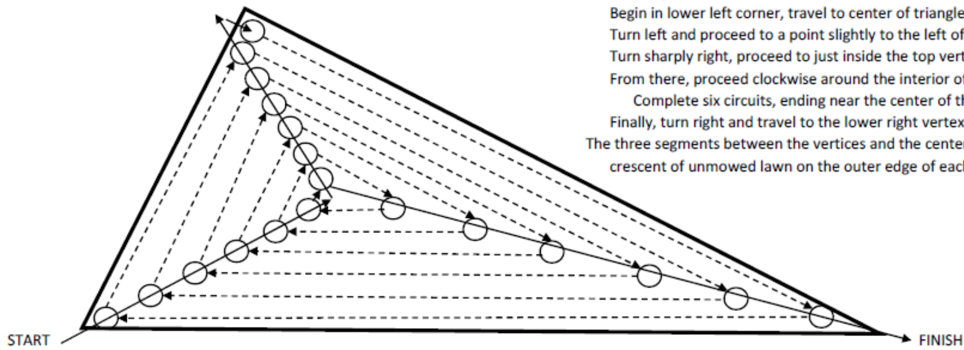
Turn sharply right, proceed to just inside the top vertex

From there, proceed clockwise around the interior of the triangle

Complete six circuits, ending near the center of the triangle

Finally, turn right and travel to the lower right vertex of the triangle

The three segments between the vertices and the center clear out the crescent of unmowed lawn on the outer edge of each mowed corner



MOWING - Spiraling Triangles
(0,0); (111.8,0); (22.4,44.7)

	Start		Finish		Slope	Distance	Area	Turn at end		Seconds
	x	y	x	y			Mowed	Direction	Degrees	
Start just inside SW vertex			1.276	0.789			7.07			0.00
to <center>	1.276	0.789	29.048	17.952	0.618	32.65	97.94	L	103.28	9.56
to N thru unmowed crescents to point that intersects with line for steps 4 and 5	29.048	17.952	20.348	44.051	-3.000	27.51	82.53	R	121.72	9.46
Return to N corner of property	20.348	44.051	23.032	42.709	-0.500	3.00	9.00		no turn	0.30
to SE corner of Loop 1	23.032	42.709	105.449	1.500	-0.500	92.15	276.44	R	153.43	16.62
to SW corner of Loop 1	105.449	1.500	2.427	1.500	0.000	103.02	309.07	R	116.57	16.89
to N corner of Loop 1	2.427	1.500	21.690	40.026	2.000	43.07	129.22	R	90.00	10.31
to SE corner of Loop 2	21.690	40.026	92.741	4.500	-0.500	79.44	238.31	R	153.43	15.35
to SW corner of Loop 2	92.741	4.500	7.281	4.500	0.000	85.46	256.38	R	116.57	15.14
to N corner of Loop 2	7.281	4.500	23.032	36.001	2.000	35.22	105.66	R	90.00	9.52
to SE corner of Loop 3	23.032	36.001	80.033	7.500	-0.500	63.73	191.19	R	153.43	13.78
to SW corner of Loop 3	80.033	7.500	12.135	7.500	0.000	67.90	203.69	R	116.57	13.38
to N corner of Loop 3	12.135	7.500	24.373	31.976	2.000	27.36	82.09	R	90.00	8.74
to SE corner of Loop 4	24.373	31.976	67.325	10.500	-0.500	48.02	144.06	R	153.43	12.21
to SW corner of Loop 4	67.325	10.500	16.989	10.500	0.000	50.34	151.01	R	116.57	11.62
to N corner of Loop 4	16.989	10.500	25.715	27.951	2.000	19.51	58.53	R	90.00	7.95
to SE corner of Loop 5	25.715	27.951	54.616	13.500	-0.500	32.31	96.94	R	153.43	10.64
to SW corner of Loop 5	54.616	13.500	21.843	13.500	0.000	32.77	98.32	R	116.57	9.87
to N corner of Loop 5	21.843	13.500	27.056	23.926	2.000	11.66	34.97	R	90.00	7.17
to SE corner of Loop 6	27.056	23.926	41.908	16.500	-0.500	16.60	49.81	R	153.43	9.07
to SW corner of Loop 6	41.908	16.500	26.698	16.500	0.000	15.21	45.63	R	116.57	8.11
to N corner of Loop 6	26.698	16.500	28.398	19.901	2.000	3.80	11.41	R	90.00	6.38
to just inside SE corner of property thru unmowed center triangle and unmowed SE crescents	28.398	19.901	110.344	0.345	-0.239	84.25	252.74			8.42
Total						974.98		20		230.50
Seconds						97.50		80.00		177.50

Wide Trapezoid

The wide trapezoid has a base 300 feet wide, a "roof" that is 100 feet wide, an altitude of 50 feet, and sides of 111.8 feet at a slope of .5. With the extreme difference between the 300-foot base, and the 50-foot altitude, the wide trapezoid is ideally suited for the Shuttle strategy. Based on the similarity of the trapezoid dimensions to those of the rectangle, we know that we will require 17 horizontal mowing strips, and that these 17 passes of the mower will leave us with 1 excess foot that we can allocate to producing time-saving overlaps in the more refined Shuttle strategies.

The basic Shuttle strategy for the wide trapezoid starts with a 300-foot mowing strip along the base, and the subsequent mowing strips, positioned at 3-foot intervals, each decrease in length by 12 feet. The final mowing strip is 108 feet. Each turn involves a shift of 6.71 feet along the slope of the trapezoid's sides, and thus the time per turn is 4 to 8.67 seconds. The total mowing distance is 3468 feet, accomplished in 346.8 seconds, and the 16 turns require 64 to 138.73 seconds, for a total project time of 410.8 to 485.5 seconds.

Refinements to this basic Shuttle strategy include first placing the turns more tightly, as we did with the triangle. The turns on the right side of the trapezoid proceed similarly as they did with the right side of the triangle, since the slopes are the same. The turns on the left side of the trapezoid follow a similar pattern as on the left side of the triangle, but shift westward by 6 feet for strip 1, 12 for strip 2, 0 feet for strip 3, followed by a repeating pattern of 12, 0, 12, 0… feet shifts. (There is nothing inherently different about the left and right sides of the trapezoid; the different turn patterns are a simple consequence of the fact that we start mowing in the lower left hand corner of the trapezoid rather than the lower right hand corner.) Including the 3 foot vertical movement, the shifts during the pivot-shift-pivot turn maneuvers are 12.369 feet, so the turn times range from 4 to 9.237 seconds. (We note that is seems unrealistic to assume that the optimistic turn time for these turns is the same as in the Rectangle solution, where the total shift lengths were only 3 feet.)

The other refinement to the Shuttle strategy for the Wide Trapezoid, as with the triangle, is utilizing the 1 foot of available overlap to capture some efficiencies within the property boundaries. As noted in the triangle discussion, with the side slopes of 0.5 in the wide trapezoid, we expect that the optimization will allocate most of the overlap to the high-numbered mowing strips. Indeed, the resulting overlap allocations (feet) are as follows, shown as the amount of overlap between strip N and the next more southerly strip N-1:

Mowing Strip #	Overlap allocated (feet)	
	Western end	Eastern end
17	.85	.94
16	.11	
15		.04
14	.02	
13		.01
12	<.01	
11		<.01
10	<.01	
9		<.01
8	<.01	
7		<.01
6	<.01	
5		<.01
4	<.01	
3		<.01
2	<.01	
1	<.01	<.01

The horizontal mowing strips in this refined Shuttle strategy total to 3371 feet (337.1 seconds of cutting time), and the 16 turns (which average 12.1 feet of shift between the two pivots) add 4 to 9.21 seconds each, or 64 to 147.3 seconds in total. Thus, the total project time is 401.1 to 484.44 seconds, not much of an improvement over the basis Shuttle strategy for the wide trapezoid. 16% to 30% of the total project time is spent on turns. We did not find an alternative mowing strategy for the wide trapezoid.

Tall Trapezoid

The tall trapezoid has a base 150 feet wide, a "roof" that is 50 feet wide, an altitude of 100 feet, and sides of 111.8 feet at a slope of 2.

In the basic Shuttle strategy, we will require 34 cuts with the 3-foot lawnmower, which will leave us with 2 feet of excess mowing that we can allocate to overlapping cuts within the property boundaries. The first mowing strip is 150 feet in length, with subsequent strips decreasing by 3 feet per strip. The final mowing strip is 51 feet long. The turns involve a shift of 3.35 feet between the two pivots, for a time of 4 to 8.335 seconds per turn, 132 to 275.1 seconds in total. The total project time is 473.7 to 616.8 seconds.

The refinements to the basic Shuttle strategy, as with the wide trapezoid, are more stringent placement of the turns, and utilizing the available 2 feet of excess mowing to gain some efficiencies within the property lines. With the steeper slopes on the sides of the tall trapezoid, the allocation of the overlap, while still decidedly concentrated at the upper end of the trapezoid, is not as extremely focused on the high-numbered mowing stripes as is the case with the wide trapezoid. The overlap allocations (feet) are as follows, shown as the amount of overlap (feet) between strip N and the next more southerly strip N-1:

Mowing Strip #	Overlap allocated (feet)	
	Western end	Eastern end
34	1.26	1.02
33		.54
32	.36	
31		.18
30	.14	
29		.09
28	.07	
27		.05
26	.04	
25		.03
24	.03	
23		.02
22	.02	
21		.02
20	.01	
19		.01
18	.01	
17		.01
2 thru 16	<.01 per even-numbered strip	<.01 per odd-numbered strip
1	<.01	<.01

With this usage of the overlap resource, the Shuttle strategy project time is reduced to 467.34 to 612.55 seconds, depending on the time required for the U-turns.

We found another strategy that is competitive with the Shuttle strategy for the tall trapezoid, which is a variation on the Spiral strategy. The best Spiral strategy that we found for the tall trapezoid has a surprising feature: the sides of the trapezoid are notionally extended upward to where they meet 150 above the center of the trapezoid base. Within that hyper-extended triangle, the Spiral strategy is implemented in essentially the same way as we did for the Triangle challenge earlier. Although it seems inherently wasteful to mow 1250 square feet of the neighbor's lawn, doing so can be faster than making the extra turns associated with Spiraling within the Trapezoid, and avoids creating a fourth diagonal of unmowed crescents.

Spiraling in the hyper-extended triangle, the strategy proceeds through 15 nested triangles, the first of which has a mowing distance (for the center of the cutting blade) of 469.70 feet (which is 15.71 feet less than the perimeter of the hyper-extended triangle). The subsequent nested triangles decrease in travel distance by 31.42 feet per cycle. The final (15^{th}) triangle has a mowing distance of 29.87 feet, and leaves an unmowed interior triangle with a base of 5.32 and an altitude of 5.32. This interior triangle is small enough to be cleared out by the three supplemental cleanup cuts that extend from the triangle vertices to the center of the triangle (and which require 228.38 additional feet of cutting, and 3 extra simple pivot turns in addition to the 45 simple pivot turns we used in the triangles). All told, the mower travels 3975.18 feet, and the project consumes 589.5 seconds, including turn time.

Further examination of this strategy reveals an opportunity for a small improvement: During the first three cycles of the triangle, the journey up to the top of the triangle's peak above the top of the tall trapezoid consumes considerably more distance than cutting across the top of the tall trapezoid. On the first cycle, for example, this journey is 104 feet, compared with 47 feet across the top of the tall trapezoid, a difference of 57 feet, or 5.7 seconds – more than the time that would be consumed by the extra pivot turn required to take the short cut across the top of the trapezoid. To a lesser degree, we find the same issue on the second and third cycles. By taking the shortcut on these three cycles, we save 15.4 seconds of cutting time, and incur an extra 12 seconds of turn time with the three extra simple pivot turns. (Note that in taking the shortcut, we only cut 1.67 feet of our property's fresh grass width along each of the three upper horizontal journeys, in order to avoid creating a new set of uncut crescents at the outer edge of our turns, which would require additional repair cuts as an additional step.) These three shortcuts reduce our project time to 586.1 seconds.

This Spiral strategy using the hyperextended triangle will outperform the Shuttle strategy if the Shuttle strategy U-turn time is 7.6 seconds or more. Recall that the U-turns in this Shuttle strategy require a pivot turn, a short shift, and another pivot turn; the Spiral Strategy requires only simple pivot turns (4 seconds).

Recap

Now we can compare all these results:

Shape	Area (sq. ft.)	Lower Bound Time (sec.)	Practical Time Range (sec.)	Efficiency Range L.B. / Pract. Times	Extra Time Range Pract. - L.B. Times (sec.)
Rectangle	5,000	166	233 - 301	71% - 55%	67 - 135
Triangle	2,500	83	139 - 207	60% - 40%	56 - 124
Parallelograms					
-long side attach	10,000	333	401 - 484	83% - 69%	68 - 151
-short side attach	10,000	333	467 - 613	71% - 54%	134 - 280