AR Puzzlement – June 2020
Straightedge and Compass Geometry
Conger submission – June 7, 2020

PART ONE

From basic geometry, we know that:

a. Angle DAB + Angle DBA + Angle ADB = 180 degrees
b. Angle ADC + Angle ADB = 180 degrees
c. Angle ACD + Angle ADC + Angle CAD = 180 degrees
d. Angle DAB + Angle EAC + Angle CAD = 180 degrees
e. Angle DBA = Angle ABC

We also know that line segments AC, AD, and BD all have lengths equal to each other and equal to the radius of the circle. Therefore, triangles CAD and ADB are isosceles, and thus

f. Angle DAB = Angle DBA
g. Angle ACD = Angle ADC

Combining (b) and (a) we get

Angle ADC = Angle DAB + Angle DBA

And then using substitutions from (e) and (f) we can restate this as

h. Angle ADC = 2 x Angle ABC

Similarly combining (c) and (d) we get

Angle ACD + Angle ADC = Angle DAB + Angle EAC

And then using substitutions from (e), (f) and (g) we can restate this as

i. 2 x Angle ADC = Angle ABC + Angle EAC

Finally, combining (h) and (i) we get

2 x (2 x Angle ABC) = Angle ABC + Angle EAC
3 x Angle ABC = Angle EAC
r = (Angle ABC) / (Angle EAC) = (1/3)

PART TWO

I am familiar with the proof that there is no general method to trisect an angle with a straightedge and compass, and I recall spending a lot of time in 8^{th} grade geometry trying to find a general method (unsuccessfully, of course).

As part of that journey, I did learn that I could get an approximation as close as desired to trisecting an angle by merely bisecting it again and again using the familiar ruler and compass method. So with five sequential bisections, I could get 11/32 of the original angle; with six sequential bisections, I could get 21/64 of the original angle; and with seven sequential bisections, I could get 43/128 (33.6%) of the original angle. Conceptually very easy, though of course difficult to achieve the mechanical precision in real life.

I have not encountered the diagram in this puzzle before, so I was delighted to find that it trisects the angle and decided to try reverse engineering that diagram with the angle XYZ, as illustrated in the attached diagram. (I have reversed the direction of angle XYZ so as to align with the other diagram in the puzzle.) Of course, this "solution" doesn't satisfy the problem's requirement about using only the simple straightedge and ruler, but it is pretty good….

Note that if the angle XYZ is greater than 135 degrees, we first will need to bisect (or quadrisect) that angle; then trisect the created angle; and then double or quadruple the angle that results from the trisection.

Steps:

1. Extend line segment ZY out beyond point Y to a distance at least two or three times the length of ZY. Select an arbitrary point out near the extended end of the segment and label that point W.
2. Using basic geometry, bisect the line segment YW and label the point of bisection as V. At point V, install a line segment that is perpendicular to line segment YW, and extends above the line segment YW. Use the label U to label a point at the upper end of the line segment. Any point on the plane that is equidistant from points Y and W (similarly to how point D is equidistant from points A and B in the original diagram) will fall on line UV.
3. Using a variety of radii (call them R1, R2, R3, etc), use the compass to draw circular arcs centered on point Y and intersecting line XY at points we will label S1, S2, S3, etc; and draw circular arcs centered on point Y and intersecting line UV at points we will label Q1, Q2, Q3, etc.
4. Now, use the straightedge to determine whether the points S1, Q1 and W all fall on a straight line. If so, then R1 is the radius of the desired circle and angle S1-W-Y is one-third of angle XYZ.
5. If R1 is not the desired radius, try R2 or R3 or test some additional Radius values until finding a Radius value Rx such that points Sx, Qx and W all fall on a straight line. Then angle Sx-W-Y is one-third of angle XYZ

I then proceeded to the internet and discovered another approach to reverse-engineering this same diagram. In this other approach (for which I can take no credit), also extend the line segment ZY but do not label the point that is out on the extension. Now, draw a circle of arbitrary radius R, centered on point Y, and label as X0 the point where the circle crosses line XY. Then, mark two points <u>on the straightedge itself</u> that are separated by distance = R, and label those two points on the straightedge as Q and W. Now, slide the straightedge around on the paper, while always maintaining that (a) some point on the straightedge always is aligned with point X0; and (b) the point marked Q on the straightedge is always on the circle. When the point W on the straightedge is on line ZY while conditions (a) and (b) also have been met, then mark that point as W. At the same time, transcribe point Q from the straightedge to the point on the circle that corresponds to point Q, i.e., label that point on the circle as Q. I tried this method also, and found it a bit easier to use than my solution.