Solve This

It's a Puzzlement by Jon Evans

The Length of Days and Nights

In this puzzle, at location A when the time is halfway between the Spring Equinox and the Summer Solstice the day is about 14 hours long. About long will the night be at A during the during the Summer Solstice? What about during the Winter Solstice?

Bob Conger submitted the following extremely detailed solution treatise:

Results:

- The earth's axis tilts 17.041 degrees towards the sun on the Puzzle Date (compared with 0 degrees towards the sun on the spring equinox and 23.436 degrees towards the sun on the summer solstice). Further discussion of this element later.
- Location A is positioned at latitude 38.067 degrees N or S, any longitude, a result that is directly depended on the angle of the axis, above.

LENGTH OF DAY					
TIME OF YEAR					
Label ✍	Spring Equinox	Puzzle Date: Mid between Spring Equ'x & Summer Sol	Summer Solstice	Autumn Equinox	Winter Solstice
CONGER SOLUTION (HOURS:MINUTES of DAYLIGHT)					
Day in the Simple Solar System (no refraction & sun radius = earth radius)	2 x 6:00 hours	2 x 6.93 hours	2 x 7.32 hours	2 x 6:00 hours	2 x 4.68 hours
Effects of Refraction and Sun Radius, morning and eve	2 x 4.11 minutes	2 x 4.44 minutes	2 x 4.78 minutes	2 x 4.11 minutes	2x 4.75 minutes
Conger Estimates (hours:minutes)					
Length of Day	12:08	14:00	14:48	12:08	9:31
Length of Night	11:52	10:00	09:12	11:52	14:29
COMPARISON TO PUBLISHED LENGTH OF DAY FOR LOCATION 38.067 N 0.0 E					
Date/Time (UTC) in 2022 (*Note 1*)	Mar 20 3:33 pm	May 6 0:23 am	June 21 09:13 am	Sept 23 01:03 am	Dec 21 9:48 pm
Length of Day Published Data For Length of Calendar Day	12:09 Mar 20 – 12:08 Mar 21 – 12:10	13:56 (*Note 2*) May 5 – 13:55 May 6 – 13:57 May 7 – 13:59 May 8 – 14:01	14:49 Jun 20 – 14:49 Jun 21 – 14.49	12:08 Sept 22 – 12:09 Sept 23 – 12:07	9:31 Dec 21 – 9:31 Dec 22 – 9:31

- *Note 1: Published length of day is from www.timeanddate.com/sun. When the event occurs late in the evening or early in the morning (UTC time), I use the average of the two bracketing dates.*
- *Note 2: See later discussion of my misestimation of the length of day on the Puzzle Day, which I suspect is driven by my approximation of a perfectly circular orbit around the sun, rather than the actual elliptical orbit.*

Solution approach:

- Step 1: Define a Simple Solar System to keep the solution manageable.

- Step 2: Calculate magnitude of the tilt of the earth's axis towards the sun when the time is halfway between the spring equinox and the summer solstice.

- Step 3: Work out the geometry and trigonometry in the Simple Solar System (including the radius of the sun being the same as earth's radius, and no atmospheric refraction) to determine the location of the sun at sunrise and sunset (i.e., what location on earth is pointing directly towards the center of the sun) on the relevant dates specified in the puzzle. This step also readily yields the latitude of Location A.

- Step 4: Adjust for the extra minutes of daylight (morning and evening) due to the refraction of the sun's rays, and the size of the sun relative to the earth. The extra minutes might vary depending on what date we are considering

Note: The work in Step 3 depends on the calculation of extra minutes of daylight in Step 4. But the work in Step 4 also depends on the results of Step 3. I found it easiest to resolve this dilemma arithmetically by iterating Steps 3 and 4 a few times. Since Step 4 produces a small answer for the puzzle's scenarios, and the calculation in this step is not particularly sensitive to small changes in the results of Step 3, the results of Steps 3 and 4 stabilize in a few iterations.

Step 1: Define a Simple Solar System (see Sketch #1 and Sketch #3)

In order to keep the geometry manageable, and in the spirit of the "approximate" perspective urged in the statement of the puzzle, I assumed:

- Earth and Sun are perfectly spherical, and the view of the sunset is from exactly at the sea level of the earth
- Earth's orbit is perfectly circular, and the Sun is at the exact center of that circle. This simplification does create a result that does not comport with empirical reality, as discussed in the last section of this document.
- A "year" means one complete earth orbit of the Sun.
- Tilt of earth's axis towards (or away from) the Sun is consistent throughout each day, and then at midnight it shifts a bit for the next day. In reality, of course, it shifts every moment as the earth progresses around the sun.

I used approximate distances and lengths, but I note that these don't affect my solution as long as I use them consistently:

- Earth radius, 4000 miles
- Sun radius, 432,000 miles
- Radius of Earth orbit, 93 million miles

In addition, for Step 3 of my solution (see Sketch #1), I assumed:

- Sun's radius is 4000 miles, the same as Earth's

- No atmospheric refraction of sunlight

I incorporated the effects of refraction and Sun's larger radius in Step 4 of my solution.

I solved the problem for the case where Location A is known to be on the zero meridian and is north of the equator, as this kept the notation and explanation less confusing. However, my "length of day" results would be the same for Location A being at any longitude, and either north of south of the equator.

For much of my explanation, I adopt the language of a geocentric view of the universe. The "location of the sun" means the point on the surface of the earth that is directly pointing towards the sun at a particular moment (i.e., the point on the surface of the earth that is in on the line segment connecting the center of the earth to the center of the sun).

"Puzzle Date" means the moment halfway along the time from the Spring equinox to the Summer Solstice, i.e., the date specified in the puzzle.

"Observed Day" or "Observed Daylight Hours" means the elapsed time from the moment any portion of the sun is first seen from an observation point on earth (sunrise) to the last moment any portion of the sun is seen at the end of the day (sunset), including the day-lengthening effects of refraction and of the size of the sun.

A "day in the Simple Solar System" means the length that an Observed Day would have if the sun's radius were equal to the earth's radius, and if there were no atmospheric refraction of the sun's rays.

Step 2: Tilt of the earth's axis on the Observation Day

- The earth's axis is tilted at 23.436 degrees away from a line that is perpendicular to the plane of the earth's orbit (plus or minus a little wobble and gradual progression). Source: published info, various sources agree closely enough for the puzzle.
- If we look down at the solar system (from above) at the time of the spring equinox, the earth's axis appears to point 90 degrees away from the sun; from the same vantage point at the summer solstice, the earth's axis appears to point directly towards the sun.
- A significant assumption in my solution is: at the halfway time between those two moments, the earth's axis, when viewed from a vantage above the solar system, appears to point at an angle 45 degrees away from the sun. This is a solid assumption with a perfectly circular orbit (i.e., in my Simple Solar System), but almost certainly incorrect for the elliptical orbit of the earth in the real solar system, since the speed of the earth's transit varies over the course of the year in its elliptical orbit. See Sketch #1. To illustrate, the dates/times of the equinoxes and solstices do not divide the year into four equal lengths.
- With some basic geometry and trigonometry, as set forth in Sketch #2, I calculate that the earth's axis is tilted 17.041 degrees towards the sun on the Puzzle Date in the Simple Solar System.

Step 3: Geometry of the sunrise and sunset locations in the Simple Solar System (see Sketch #5)

- Mentally, I have positioned myself relative to the earth so that the point 90 degrees W, 0 degrees north (the Galapagos Islands, about 500 miles west off the coast of Ecuador, a place I've always wanted to visit) is directly facing me, and I am aligned so that the earth's axis is directly vertical in my perspective. Location A of the puzzle (on the zero meridian) is on the far right edge of the globe, somewhere north of the equator, and the international date line is on the far left edge of the globe. I am very far away from the earth, so the lines of latitude all appear as horizontal lines.

- Since geometry on a sphere is pretty complicated, I also utilize an XYZ Cartesian labeling of locations. The x-axis penetrates the equator at longitudes 0 and 180 degrees. The y-axis penetrates the equator at longitudes 90 west and 90 east. The z-axis coincides with the earth's axis. As I describe points on the surface of the earth, I also map them on the XZ plane, and on the XY plane (Sketch #5).
- On a given date in my Simple Solar System, the position of the sun travels along one of the lines of latitude. On the Puzzle Date, it travels along 17.041 degrees north. On the solstices, it travels along 23.44 degrees north (summer solstice) or 23.44 degrees south (winter solstice). On the equinoxes, it travels along the equator.
- Location A observes 14 hours of daylight on Puzzle Day. Subtracting 4.44 minutes from either end of the Puzzle Day for the effects of the sun's large radius and atmospheric refraction of sunlight (derived in Step 4), Location A would experience 6.93 hours of daylight before noon and 6.93 hours of daylight after noon on Puzzle Date in the Simple Solar System, which is equivalent to 103.89 degrees of rotation. A little bit of math yields that at sunset on Puzzle date for Location A in the Simple Solar System, the sun is over a point 1172 miles north of the equator and 918 miles west of the earth axis as seen on the XZ plane. See Sketch #5 and Calculation exhibit.
- Throughout the year, at any sunset (or sunrise) at Location A in the Simple Solar System, the sun's location at sunrise/sunset is on the great circle formed by the intersection of (a) the earth's surface with (b) a plane that goes through the center of the earth ("Point C"), and is perpendicular to a line segment connecting Location A to Point C. See Sketch #3. The y-axis is contained in this plane, and this plane thus is perpendicular to the XZ plane. I observe the nice result that Z divided by X, regardless of date, is the same for Location A for all sunset and all sunrise positions of the sun (Simple Solar System). (Exception noted when Z and X are both zero.) A small bit of math yields the X-Z coordinates of sunset on the solstice dates requested in the puzzle. On a given day, the X and Z values at sunset are the same as they were at sunrise. A bit more math yields the degrees of earth rotation between noon and sunset (and same from sunrise to noon), and thus the hours of daylight in the Simple Solar System. See Sketch #5 and Calculation Exhibit.
- Finally, add in the additional minutes of daylight due to the large radius of the sun, and the effects of refraction, as derived in Step 4. Final results shown on the Calculation Exhibit, as well as on the first page of this text.
- Based on the constant relationship of Z and X, we can calculate the angle between the equatorial plane and the plane that contains all the sunrise and sunset points for Location A in the Simple Solar System (51.93 degrees). Then, since this plane is perpendicular to line segment AC, the latitude of Location A is 38.07 degrees.

Step 4: Effects of Large Sun Radius and Refraction of Sunlight (see Sketch 3)

- A quick bit of research yields that the effect of refraction on the observed time of sunrise or sunset is about 0.57 degrees (though varies depending on temperature), and a little math yields the result that the radius of the sun in excess of the earth's radius causes the real sun to poke above the horizon about 0.26 degrees more than the sun in the Simple Solar System.
- However, I opted to use an empirical measure based on observing (in published data) that a point on the equator experiences 12:06:28.5 (hrs:mins:secs) of daylight on the equinoxes. The extra 3.24 minutes of sunlight at sunrise and sunset translate to approximately 0.81 degrees of rotation, or 56.5 miles along the earth's surface.

- At points such as Location A, it takes more than 3.24 minutes for the sun to set the additional 0.8 degrees because the sun's path in the sky is at an angle to the horizon. As a first estimate, I assume this angle of the last few minutes of the sun's journey to the horizon equals the angle of the sunrise/sunset plane from the equatorial plane (51.93 degrees, as described in Step 3), and that the sun must travel 3.24 / sin (51.93 degrees) = 4.11 extra minutes to sunset (and from sunrise), which is easily translated into degrees of rotation.
- The actual angle of the sun's path to the horizon varies depending on the latitude at which the sun is traveling, and how far around the earth the sunrise and sunset points are located from Location A. I wasn't sure about the magnitude of this effect, but I was having fun, so I fine-tuned this estimate for each of the Puzzle dates, as follows (and as illustrated in Sketch #4). I did the calculations for sunset, and the result applies equally to sunrise.
    - Guess the extra minutes ("M") of sunset daylight for Location A on Puzzle Day, and translate the extra minutes into degrees of earth rotation ("D").
    - Use M in Step 3 to adjust the daylight hours (eliminating the extra minutes resulting from refraction and sun size) for Location A on Puzzle Day, and proceed through Step 3 to calculate the latitude and longitude of Location A, and the location S of the sun at sunset (Simple Solar System). The distance from A to S is 90 degrees around the earth, so = earth radius x PI / 2 = 6283 miles along the surface of the world.
    - Move D degrees west of location S on the same latitude line to get location T of the sun when it appears to set from Location A, including the effects of refraction and sun size. Note that distance A to T along the earth's surface is further than A to S along the earth's surface.
    - Using the latitude and longitude of A and T, calculate the distance along the earth's surface (Simple Solar System) from A to T. I was delighted to learn that there is a simple method (Haversine Formula) to perform this calculation.
    - If distance A to T is 56.5 miles further than distance A to S, I'm done. Otherwise, adjust the guess of extra minutes of daylight, and repeat the calculations in Step 3 until I find a guess that satisfies (distance A to T) = (distance A to S) + 56.5 miles. Given that the extra minutes are a small fraction of the total hours of daylight, the guesses and calculations proceed quickly to convergence.
    - Perform similar calculations for the other relevant dates. The calculations for other dates do not alter the determination of the latitude of Location A, so they converge very quickly. As it turns out, the extra minutes of daylight at sunset do vary by date, but the results for the equinox align with my initial estimate (4.11 minutes). At the summer solstice, the result is 4.78 minutes. (Same results at sunrise.)

Empirical Check

- Having determined the latitude of Location A (and using longitude zero), I looked up the daylight hours for the location for the various puzzle dates (in 2022). The empirical hours are shown in the table on page 1 of this document.
- My estimated daylight hours for the summer solstice and winter solstice, and also for the equinoxes, align very closely with the empirical results. Yay!
- Sadly, my daylight hours for the Puzzle Date are off by several minutes from the empirical hours for Location A (14:00 versus 13:56). I also checked with Location A being in the southern hemisphere (and reversing the seasons so the Puzzle Date is now early morning on November 6), and found a slightly larger discrepancy (14:00 versus 13:52).
- I hypothesize that in the real world the tilt of the axis towards the sun on the halfway date between spring equinox and summer solstice is not the same as it is in my Simple Solar System. As noted earlier, the speed of travel of the earth varies over the course of the year in its actual

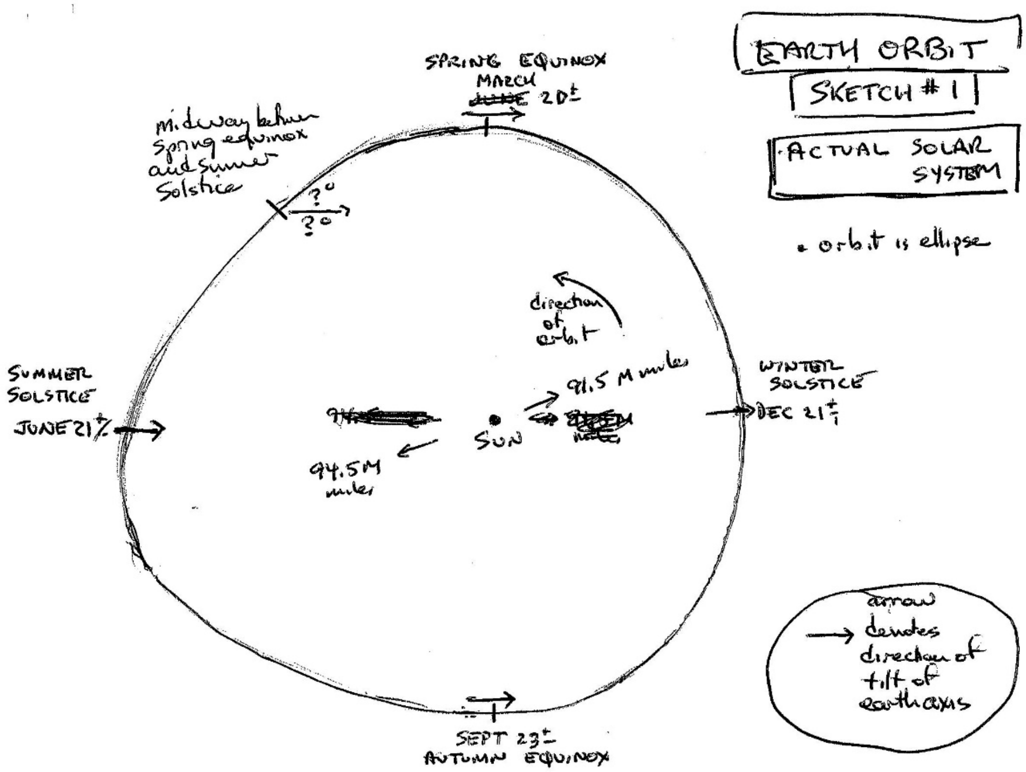
elliptical orbit and the sun is not at the center of the ellipse, and thus it seems likely that halfway through the time between the spring equinox and summer solstice would not translate to the earth having transited half of the degrees of arc between those two moments. However, I already have spent more time than I should have playing with this entertaining puzzle, and I'm pretty sure that working out the angles on the ellipse of the earth's actual orbital path is beyond the scope of what you envisioned in the puzzle.

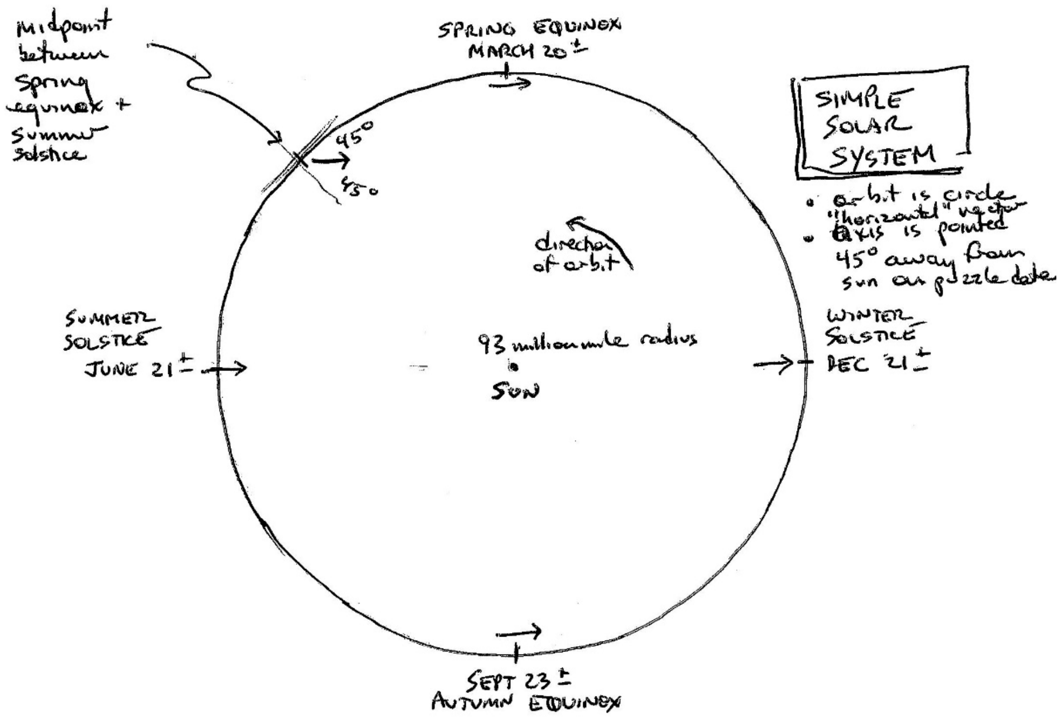
Reverse Engineering

- With a little exploring, I found that at latitude 39.00 degrees north, the empirical daylight is 14:00 hours on the Puzzle Date. Then, with reverse engineering through my solution, the angle of the earth's axis towards the sun on that date needs to be 16.5 degrees. This angle then produces the following comparison of puzzle solutions and observations regarding length of daylight (HH:MM:SS):

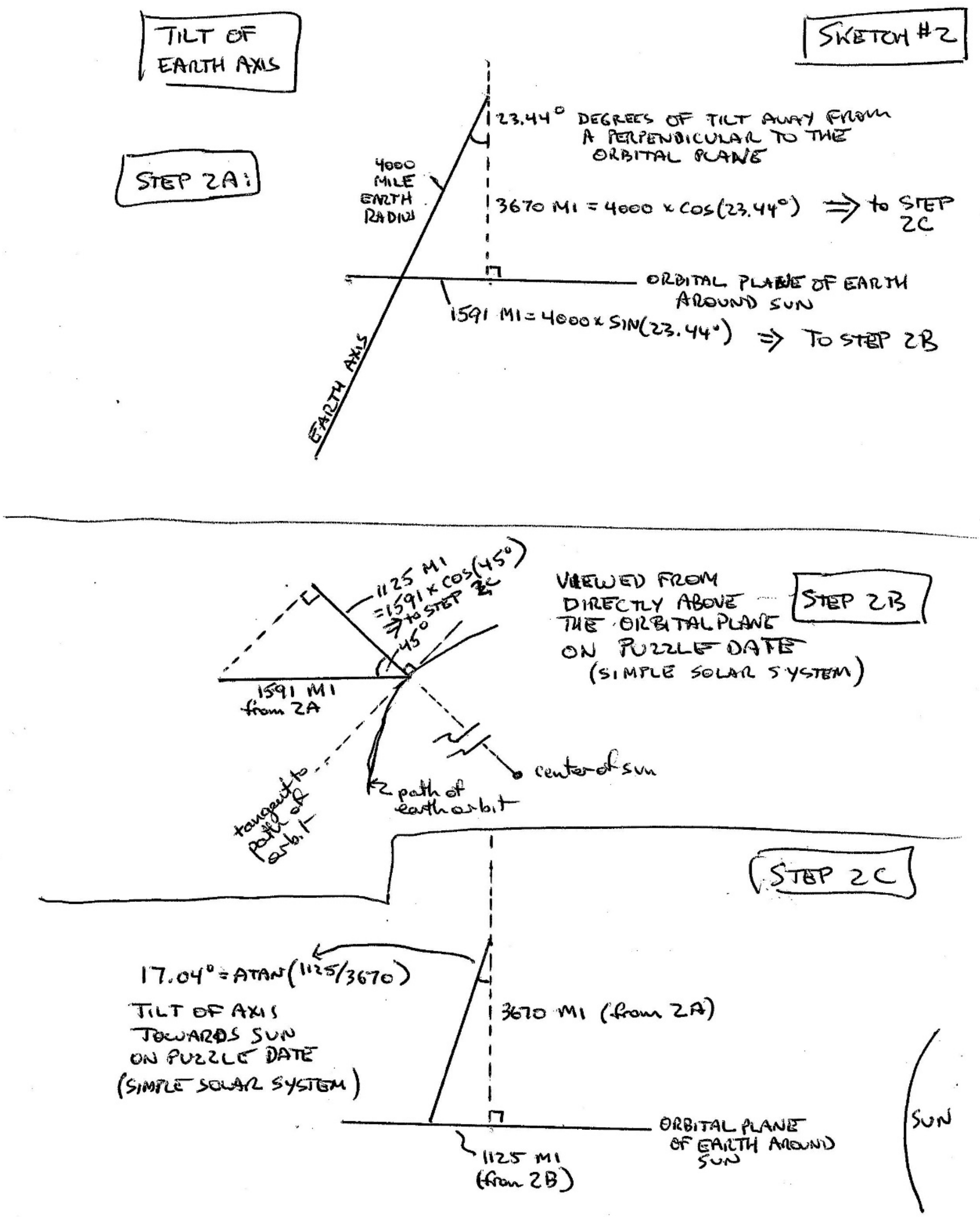
	Spring Equinox	Puzzle Date	Summer Solstice	Autumn Equinox	Winter Solstice
My calc @16.5	12:08:20	14:00:00	14:54:08	12:08:20	9:25:16
Empirical	12:08:30	13:59:59	14:54:29	12:08:22	9:25:47
Difference (secs)	- 0:10	+0:01	-0:21	-.02	-0:31

(Empirical figures are based on my interpolation of two bracketing dates.) This is pretty close! I certainly don't count this as a solution, since I was only able to arrive at the angle by already having looked up the answers to the puzzle's daylight hours question. But it does increase my confidence in my methodology, and my confidence that the flaw in my solution is the use of the Simple Solar System to estimate the tilt angle of the earth's axis on the Puzzle Date.

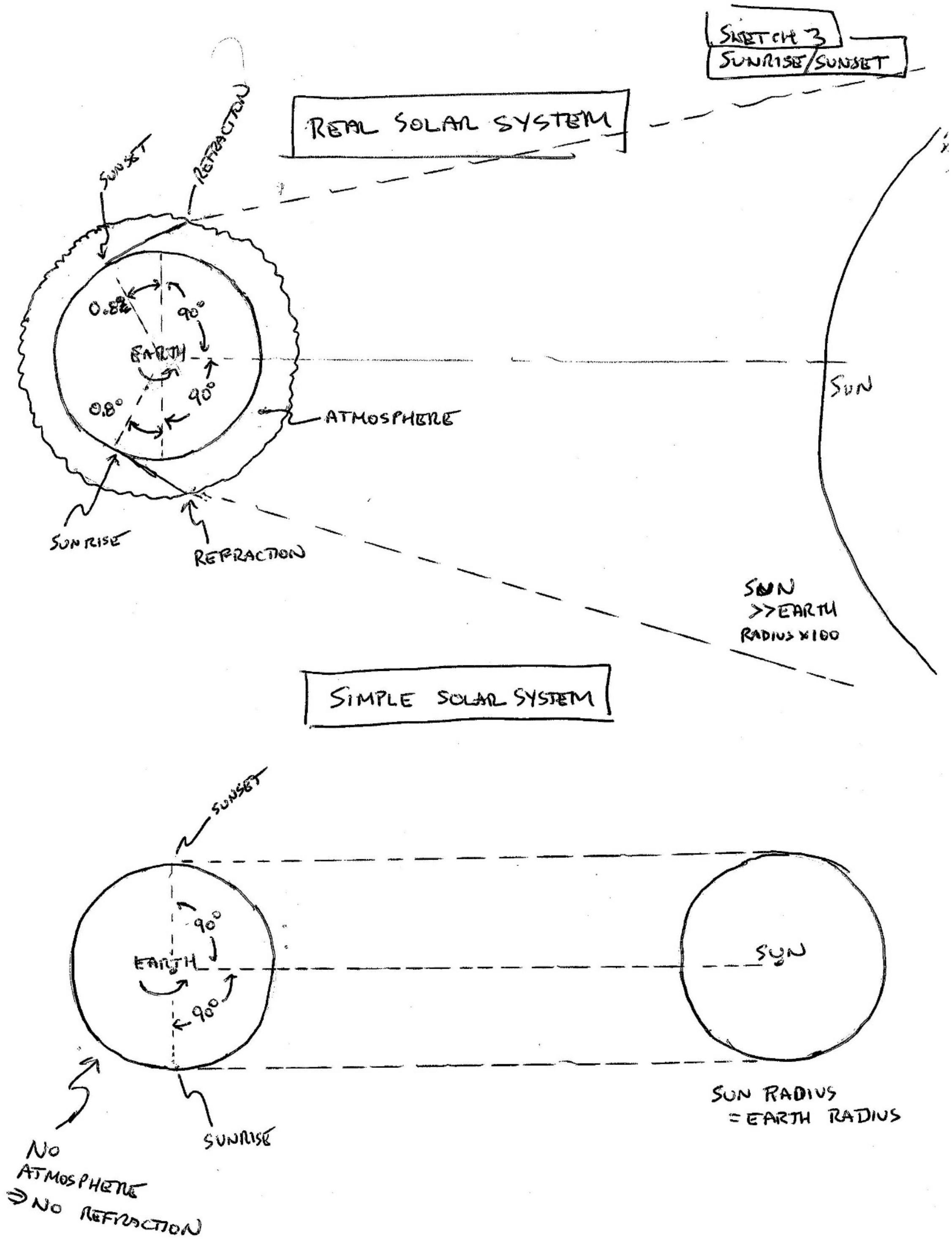

EARTH ORBIT
SKETCH #1
ACTUAL SOLAR SYSTEM
• orbit is ellipse
SPRING EQUINOX
MARCH ~~JUNE~~ 20±
midway between spring equinox and summer solstice
?°
?°
direction of orbit
SUMMER SOLSTICE
JUNE 21±
91.5 M miles
SUN
94.5M miles
WINTER SOLSTICE
DEC 21±
arrow denotes direction of tilt of earth axis
SEPT 23±
AUTUMN EQUINOX


midpoint between spring equinox + summer solstice
SPRING EQUINOX
MARCH 20±
45°
45°
SIMPLE SOLAR SYSTEM
• orbit is circle
• "horizontal" vector axis is pointed 45° away from sun on puzzle date
direction of orbit
SUMMER SOLSTICE
JUNE 21±
93 million mile radius
SUN
WINTER SOLSTICE
DEC 21±
SEPT 23±
AUTUMN EQUINOX

Sketch #1


TILT OF EARTH AXIS
SKETCH #2
STEP 2A:
23.44° DEGREES OF TILT AWAY FROM A PERPENDICULAR TO THE ORBITAL PLANE
4000 MILE EARTH RADIUS
3670 MI = 4000 x COS(23.44°) ⇒ to STEP 2C
ORBITAL PLANE OF EARTH AROUND SUN
1591 MI = 4000 x SIN(23.44°) ⇒ TO STEP 2B
EARTH AXIS
1125 MI = 1591 x COS(45°) ⇒ to STEP 2C
45°
VIEWED FROM DIRECTLY ABOVE THE ORBITAL PLANE ON PUZZLE DATE (SIMPLE SOLAR SYSTEM)
STEP 2B
1591 MI from 2A
center of sun
path of earth orbit
tangent to path of orbit
STEP 2C
17.04° = ATAN(1125/3670)
TILT OF AXIS TOWARDS SUN ON PUZZLE DATE (SIMPLE SOLAR SYSTEM)
3670 MI (from 2A)
ORBITAL PLANE OF EARTH AROUND SUN
1125 MI (from 2B)
SUN

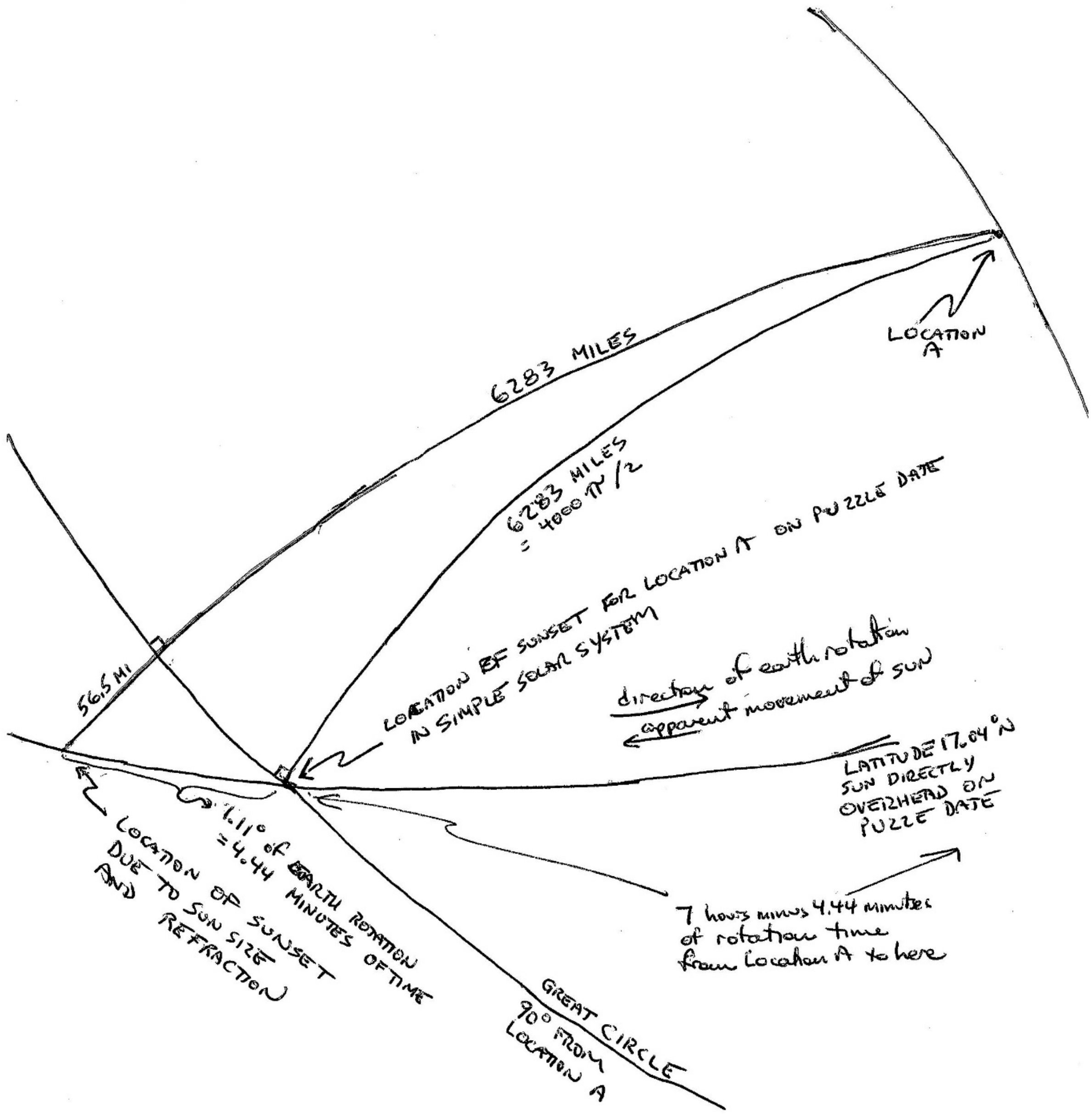
Sketch #2


SKETCH 3
SUNRISE/SUNSET
REAL SOLAR SYSTEM
SUNSET
REFRACTION
0.8°
90°
EARTH
90°
0.8°
ATMOSPHERE
SUN
SUNRISE
REFRACTION
SUN
>>EARTH
RADIUS x100
SIMPLE SOLAR SYSTEM
SUNSET
90°
EARTH
90°
SUN
NO
ATMOSPHERE
⇒NO REFRACTION
SUNRISE
SUN RADIUS
=EARTH RADIUS

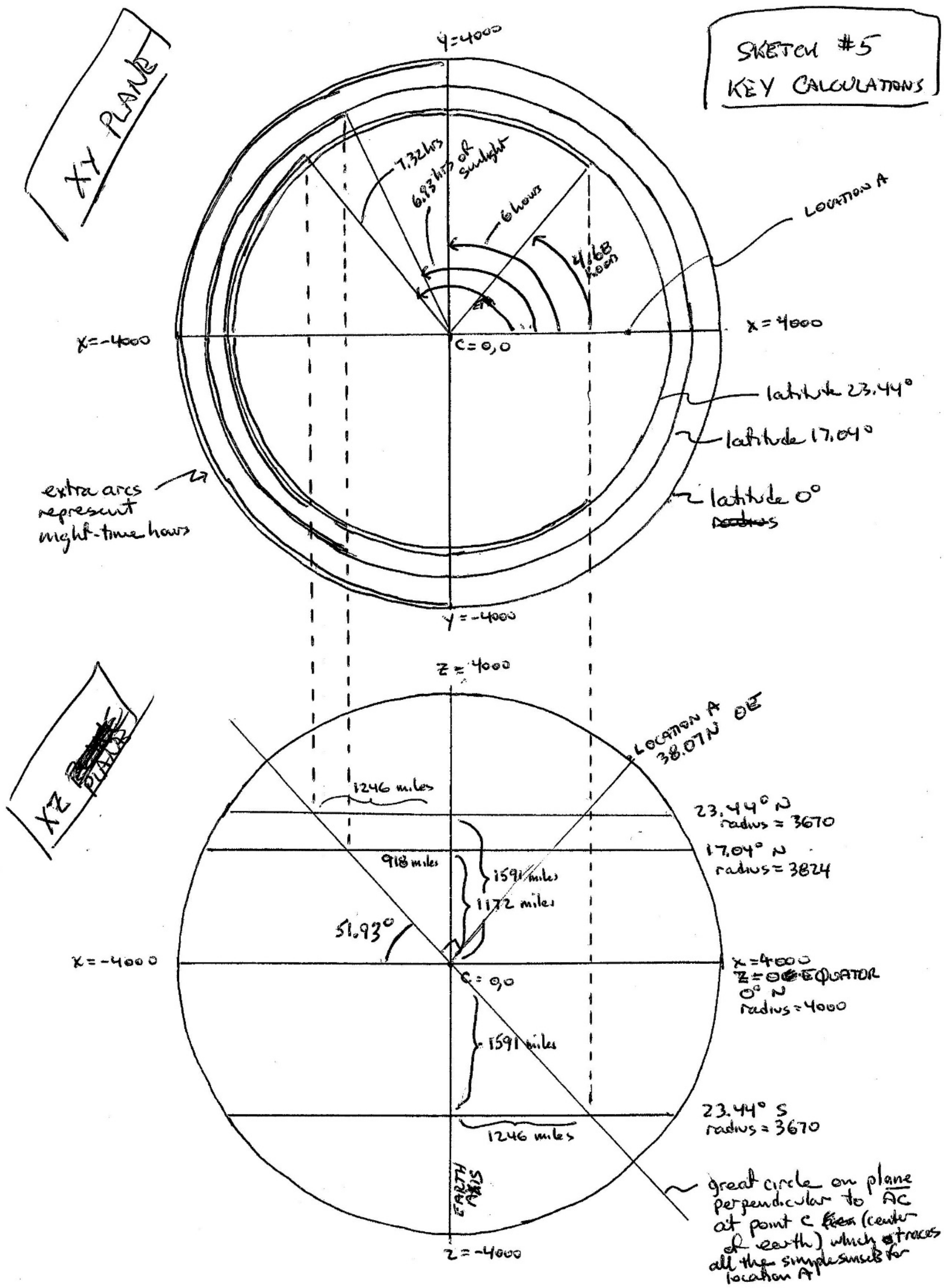
Sketch #3

SKETCH #4

ON PUZZLE DATE
HOW MUCH LATER IS SUNSET
DUE TO
SIZE OF SUN (100 x EARTH RADIUS)
AND
SUNLIGHT REFRACTION IN ATMOSPHERE
?



Sketch #4



Sketch #5

Solutions were also submitted by….(add any others received)