Actuarial Review – Puzzlement
November/December 2023 – Turning into Traffic
Submission by Bob Conger – version 3
12/17/2023

Results

Which Puzzle	Right Turn	Left Turn
Frequency of cars passing	1 per 4 seconds (1 lane must clear)	3 per 4 seconds (3 lanes must clear)
Lambda (per second) for Poisson	0.25	0.75
Likelihood of next car in < 4 seconds	63.2%	95.0%
Mean waiting time	2.9 seconds	21.4 seconds
90^{th} %-ile waiting time	7.9 seconds	50.7 seconds
99^{th} %-ile waiting time	17.3 seconds	101.4 seconds

Methodology and Discussion

For this version of my solution, I assumed we are dealing with a Poisson distribution, and used the given data (equivalent to lambda = .25 or .75 per second) to calculate the probability of no car coming within 4 seconds (37% for lambda =.25 in the "right turn" scenario where 1 relevant car passes every 4 seconds, and 5% for lambda = .75 in the "left turn" scenario where 3 relevant cars pass every 4 seconds). I then used two methods, MODEL A and MODEL B. Model B is a refinement of Model A.

In Model A, I simply used the probabilities above to calculate the probability of having to wait for no cars (=37% for lambda = .25, for example), or having to wait for exactly 1 car (= .63 x .37) or 2 cars (=.63^2 x .37) etc.. From those results, we can easily calculate the mean # cars that we have to wait for, or the 90^{th} or 99^{th} percentile results. I then convert those car numbers to time (# of seconds) by multiplying by the average arrival time of a car that arrives within the range of 0 to 4 seconds. These mean arrival times per car are 1.67 seconds for lambda=.25, and 1.12 seconds for lambda=.75. The resulting Mean wait times are 2.9 seconds (for lambda = .25); and 21.4 seconds (lambda =.75).

I was satisfied with Model A for the Mean result, but was concerned about using it for the 90^{th} and 99^{th} percentile results because the cars are passing at random moments within the 0 to 4 second range. So for example, with lambda = .25, the 90^{th} percentile number of cars we have to wait for in Model A is 5 cars (which Model A translates to 8.4 seconds, but there could be a situation with 4 cars, in which each car arrives 3.5 seconds after the prior car for a total waiting period of 14 seconds (i.e., greater than Model A's 90^{th} percentile) – or a situation with 6 cars, in which each car arrives 0.25 seconds after the prior car, for a total waiting period of 1.5 seconds (i.e., less than Model A's 90^{th} percentile).

In Model B, I incorporated the distribution of whether the cars that come along within less than 4 seconds do so in less than 1 second, 1 to 2 seconds, 2 to 3 seconds, and 3 to 4 seconds, as shown in the tables on the final two pages of this document (I thought that one-second intervals would be sufficiently granular to observe the variability, and I also could not figure out a good way to use the pure distribution). To keep the number crunching a bit simpler, I made the approximation that those cars came along at exactly 0.5 or 1.5 or 2.5 or 3.5 seconds. (I know that the average arrival time in the one-second intervals would be a little faster than that, but it was easier for me to be able to bucket things in exact half seconds. I make a corrective adjustment for this approximation at the end.) From these two elements, I could figure out the distribution of total elapsed wait time for the situation in which I have to wait for 2 cars, the distribution for 3 cars, …..and for 164 cars. Then, cross-multiplying all of these "elapsed wait time" distributions, against the distribution of the likelihood of having to wait for 0 cars, 1 car, 2 cars etc (from Model A), and summing all the probabilities associated with each possible wait time, I arrive at a probability distribution for the total elapsed wait time. Then, this probability distribution yields the mean wait time, and of course the 90th and 99th %-ile wait times.

The average arrival time for cars that arrive within 4 seconds, based on the 0.5 / 1.5 / 2.5 / 3.5 approximation differs from the average arrival time derived from the Poisson distribution, as shown below.

Lambda	Avg Poisson arrival time	Avg arrival time using 0.5 / 1.5 / 2.5 / 3.5 avg in each one-second interval	Adjusting correction per car
.25	1.672 seconds	1.693 seconds	-0.021 seconds
.75	1.124 seconds	1.186 seconds	-0.062 seconds

I therefore adjusted the Model B results by subtracting a correction factor equal to the number of passing cars (from Model A) multiplied by the correction per car shown above.

I was not surprised to find that the Mean wait times were very similar from Model A as from Model B (see tables on final two pages; the results between Model A and B actually differ by less than .005 seconds).

However, I was quite surprised to find that the Model A and B tail percentile wait times are quite similar, i.e., whether or not we incorporate dispersion of the arrival time within the 0 to 4 second window. For example, with lambda = .75,

- Model A indicates that we would have to wait for 90 cars to pass at the 99th % tile based on vehicle counts, which we translate into 101.1 seconds;
- Within Model B we can discover that a 101-second waiting period represents widely differing percentiles of the vehicles depending on the number of cars that we have to wait for; see table at the top of the following page. Note that in the 90-car scenario (which is the 99th %-ile result by vehicle count), there is a 54% chance that the 90 cars will pass within 101 seconds; This result seems to confirm the relevance of looking at the time distributions.
- But, when we probability-weight the Model B results across all the different vehicle count scenarios, the overall 99th %-ile wait time is 101.4 seconds, i.e., only marginally different from the Model A result of 101.1 seconds. The close similarity of the final Model A and Model B suggests that there is a mathematical reason lurking in the behavior of the Poisson distribution for the arrival time dispersions to average out across the whole distribution of outcomes. My math skills and energy level are not up to pursuing the derivation of that reason.

Lambda = .75		
Number of cars that we have to wait for (illustrative sample of cases)	% likelihood that this is the number of cars we will have to wait for	If we do have to wait for the number of cars, the % likelihood that they will pass by in an aggregate time of less than 101 seconds
Fewer than 27 cars	76%	100% **
70	0.14%	99.9%
80	0.08%	93%
90 (=Model A 99th %-ile)	0.05%	54%
100	0.03%	10%
110	0.02%	1%
Weighted average of all situations, from 0 cars to 164 cars*	99.98%	99.0%

*Although the table shows only a few illustrative cases, I actually performed the calculations for each case from 0 to 164 cars passing by before I can turn. I stopped calculating at 164 cars for practical reasons, but it is not material. There is only a 0.02% chance of having to wait for more than 164 cars.

**Recalling that a car must pass in less than 4 seconds in order for it to delay my vehicle's turn, the largest possible aggregate time delay for 26 or fewer passing cars is 26 x 4 = 104 seconds.

I am attaching two tables summarizing some intermediate calculations and some results.

I'd be happy to share the Excel worksheet that I used, but it is pretty ungainly. And I expect that you already have developed a more elegant solution.

AR Puzzlement Nov 2023 - Traffic delays   Conger v3 12/17/2023

Assume Poisson distribution of passing cars

1 cars per 4 seconds

0.25 Lambda (per second)

MODEL A: Based on probabilities of NUMBERS of consecutive cars arriving in less than 4 seconds. Convert to time by multiplying by MODEL B overall average arrival time for cars that arrive in less than 4 seconds

MODEL B: Starts with Model A probabilities of 0, 1, 2, etc consecutive cars arriving in less than 4 seconds. Then, for each different number of consecutive cars arriving, calculates the probabilities of the total elapsed time based on each individual car's probability of arriving in 0 to 1 second, 1 to 2 seconds, 2 to 3 seconds and 3 to 4 seconds. Combining all these probabilities, we can tabulate the overall probability of each possible waiting time.

t (secs)	Prob next car > t	Increm Prob	Balanced Likelihood	Variable Name	Selected Arvl Time (secs) MODEL B
0	1.000	0.221	0.34993	Prob05	0.5
1	0.779	0.172	0.27253	Prob15	1.5
2	0.607	0.134	0.21224	Prob25	2.5
3	0.472	0.104	0.16530	Prob35	3.5
4	0.368			ProbGT4	
Sum / Wtd Avg		0.632	1.000		1.693
					MODEL A
		Correct avg time for 0 to 4 secs			1.672

Results for Lambda = 0.25 (1 car per 4 secs)

MODEL A		MODEL B		
# cars	# secs	# secs		
		Model	Adj	
1.7	2.9	2.9	2.9	Mean
5	8.4	8.0	7.9	90th %-ile
10	16.7	17.5	17.3	99th %-ile

AR Puzzlement Nov 2023 - Traffic delays   Conger v3 12/17/2023

Assume Poisson distribution of passing cars

3 cars per 4 seconds

0.75 Lambda (per second)

MODEL A: Based on probabilities of NUMBERS of consecutive cars arriving in less than 4 seconds. Convert to time by multiplying by MODEL B overall average arrival time for cars that arrive in less than 4 seconds

MODEL B: Starts with Model A probabilities of 0, 1, 2, etc consecutive cars arriving in less than 4 seconds. Then, for each different number of consecutive cars arriving, calculates the probabilities of the total elapsed time based on each individual car's probability of arriving in 0 to 1 second, 1 to 2 seconds, 2 to 3 seconds and 3 to 4 seconds. Combining all these probabilities, we can tabulate the overall probability of each possible waiting time.

t (secs)	Prob next car > t	Increm Prob	Balanced Likelihood	Variable Name	Selected Arvl Time (secs) MODEL B
0	1.000	0.528	0.55528	Prob05	0.5
1	0.472	0.249	0.26230	Prob15	1.5
2	0.223	0.118	0.12390	Prob25	2.5
3	0.105	0.056	0.05853	Prob35	3.5
4	0.050			ProbGT4	
Sum / Wtd Avg		0.950	1.000		1.186
					MODEL A
		Correct avg time for 0 to 4 secs			1.124

Results for Lambda = 0.75 (3 car per 4 secs)

MODEL A		MODEL B		
# cars	# secs	# secs		
		Model	Adj	
19.0	21.4	22.6	21.4	Mean
45	50.6	53.5	50.7	90th %-ile
90	101.1	107.0	101.4	99th %-ile