Solve This
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We have cars arriving every 4 seconds on the average in each lane. Situations like this are usually modeled by a Poisson process – assuming that the cars' arrivals are completely independent and random. In real life, this isn't true. This would be closer to reality if the cars were arriving more rarely. The other end of the spectrum would be if the cars arrived at regular times. This is closer to traffic being very heavy. Most of the time, real life is somewhere between the two.

Let's assume that the cars in each lane follow a Poisson process with the rate λ= ¼. The spacing between two consecutive cars follows an exponential distribution with PDF:
$f(t) = 0.25e^{-0.25t}$

and CDF:
$F(t) = 1-e^{-0.25t}$

We can use the CDF to find the chance of a gap of a certain length. For a gap of 4 seconds, we first find:
$F(4) = 1-e^{-0.25(4)} = 1-e^{-1} \approx 0.63212$
which means that there's a 63.2% chance of having a car in the first four seconds. We subtract this from one to get the chance of a 4 second gap in traffic when we first arrive: 1-0.63212 = 0.36788
So 36.7% of the time, we will have to wait for zero cars to pass – and we can go immediately.

What if we have to wait for exactly one car to pass?
The probability of this happening ≈ .63212 × .36788
The average length of time that we will wait is taken from the exponential distribution as

$$E[T \mid T<4] = \frac{\int_0^4 xf(x)dx}{\int_0^4 f(x)dx} = \frac{\int_0^4 xf(x)dx}{F(4)} = \frac{(-8)e^{-1}+4}{1-e^{-1}} \approx 1.67209 \text{ seconds}$$

What if we have to wait for exactly two cars to pass?
The probability of this happening ≈ $.63212^2 \times .36788$
The average length of time that we will wait is 1.67209 seconds × 2 ≈ 3.34419

What if we have to wait for exactly n cars to pass?
The probability of this happening ≈ $.63212^n \times .36788$
The average length of time that we will wait ≈ 1.67209 seconds × n

If we take n from 0 to infinity, we can find the average wait.

$$\sum_{n=0}^{\infty}(1-e^{-1})^n(e^{-1})E[T|T<4]n$$

$$= (e^{-1})E[T|T<4]\sum_{n=0}^{\infty}(1-e^{-1})^n n$$

$$= (e^{-1})E[T|T<4]\frac{(1-e^{-1})}{[1-(1-e^{-1})]^2}$$

$=(e^{-1})\frac{(-8)e^{-1}+4}{1-e^{-1}} \times \frac{(1-e^{-1})}{(e^{-1})^2}$

$= \frac{(-8)e^{-1}+4}{e^{-1}}$

= -8 + 4e ≈ 2.87313 seconds

This is the average wait time for turning into lane one.

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If we want to turn left, we need to turn into lane three. This would mean that we would need to pass through lanes one and two. In real life, we can pass through a lane that didn't have a big enough gap for us to merge into it. But they didn't tell us how much time it takes to pass through a lane. Therefore, let's pretend that we require a 4 second gap to pass through a lane. This would mean that we have to have a four second gap in lanes one, two and three – all at the same time. This lets us simplify the problem by pretending that all cars in the first three lanes have been compressed into one lane that has three times as much traffic.
This pretend lane follows a Poisson process with the rate λ=¾. We can re-do the above work for this distribution.
The spacing between two consecutive cars follows an exponential distribution with PDF:
$g(s) = 0.75e^{-0.75s}$

and CDF:
$G(s) = 1-e^{-0.75s}$

We can use the CDF to find the chance of a gap of a certain length. For a gap of 4 seconds, we first find:
$G(4) = 1-e^{-0.75(4)} = 1-e^{-3} \approx 0.95021$
which means that there's a 95% chance of having a car in the first four seconds. We subtract this from one to get the chance of a 4 second gap in traffic when we first arrive: 1-0.95021 = 0.04979
So 5% of the time, we will have to wait for zero cars to pass – and we can go immediately.

What if we have to wait for exactly one car to pass?
The probability of this happening ≈ 0.95021 × 0.04979
The average length of time that we will wait is taken from the exponential distribution as

$E[S \mid S<4] = \frac{\int_0^4 xg(x)dx}{\int_0^4 g(x)dx} = \frac{\int_0^4 xg(x)dx}{G(4)} = \frac{\left(-\frac{16}{3}\right)e^{-3}+\frac{4}{3}}{1-e^{-3}} \approx 1.12375$ seconds

What if we have to wait for exactly two cars to pass?
The probability of this happening ≈ $0.95021^2 \times 0.04979$
The average length of time that we will wait is 1.12375 seconds × 2 ≈ 2.24750

What if we have to wait for exactly n cars to pass?
The probability of this happening ≈ $0.95021^n \times 0.04979$
The average length of time that we will wait ≈ 1.12375 seconds × n

If we take n from 0 to infinity, we can find the average wait.

$\sum_{n=0}^{\infty}(1-e^{-3})^n(e^{-3})E[S|S<4]n$

$= (e^{-3})E[S|S<4]\sum_{n=0}^{\infty}(1-e^{-3})^n n$

$= (e^{-3})E[S|S<4]\frac{(1-e^{-3})}{[1-(1-e^{-3})]^2}$

$=(e^{-3})\frac{\left(-\frac{16}{3}\right)e^{-3}+\frac{4}{3}}{1-e^{-3}}\times\frac{(1-e^{-3})}{(e^{-3})^2}$

$=\frac{\left(-\frac{16}{3}\right)e^{-3}+\frac{4}{3}}{e^{-3}}$

$= \left(-\frac{16}{3}\right)+\frac{4}{3}e^{3} \approx 21.44738$ seconds

This is the average wait time for turning into lane three.

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To find the 10% and 1% worse cases, we can simulate the traffic with an R program:

```
sumFun <- function()
{ gap <-(-log(1-runif(1))/.25)
  mySum<-0
  while (gap<4)
  { mySum<-mySum+gap
    gap <-(-log(1-runif(1))/.25)
  }
  return(mySum)
}

y <- numeric(100000)
for (i in 1:100000) y[i] <- sumFun()

mean(y)
quantile(y, probs = 0.9)
quantile(y, probs = 0.99)
```

This simulates the right hand turn into the first lane and finds the mean, 90% percentile and 99% percentile. Each run has 100,000 wait times. I ran it three times.

```
[1] 2.883373
    90%
7.98477
    99%
17.3011

[1] 2.860619
     90%
7.950186
     99%
17.10862

[1] 2.871048
     90%
7.971722
     99%
17.32575
```

We computed the mean to be 2.873 seconds.
The worst 10% case is about 8 seconds.
The worst 1% case is about 17 seconds.

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We tweak the program slightly to simulate the third lane.

```
sumFun <- function()
{ gap <-(-log(1-runif(1))/.75)
  mySum<-0
  while (gap<4)
  { mySum<-mySum+gap
    gap <-(-log(1-runif(1))/.75)
  }
  return(mySum)
}

y <- numeric(100000)
for (i in 1:100000) y[i] <- sumFun()

mean(y)
quantile(y, probs = 0.9)
quantile(y, probs = 0.99)
```

This simulates the left hand turn into the third lane and finds the mean, 90% percentile and 99% percentile. Each run has 100,000 wait times. I ran it three times.

```
[1] 21.45277
     90%
50.79433
     99%
102.5706

[1] 21.45599
     90%
50.58486
     99%
102.4809

[1] 21.40602
     90%
50.67476
    99%
101.872
```

We computed the mean to be 21.447 seconds.
The worst 10% case is about 51 seconds.
The worst 1% case is about 102 seconds.

If I had to wait 102 seconds, I would look for an alternative – like turning right and then pulling a u-turn. This does happen in real life.